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6 STRESS WAVE PROPAGATION

IN LAYERED ELASTIC CYLINDERS

10 by

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List of Symbols

a	length of first layer of bar
b	length of second layer of bar
c	acoustic velocity
$\delta(t-\tau)$	Dirac delta function (0 if $t \neq \tau$)
ϵ	normal strain
k	travel time
n	number of terms in an infinite series
p	pressure (compressive normal stress)
p_o	peak pressure
ρ	mass density of bar material
s	time parameter used with Laplace transform
σ	normal stress
t	time coordinate
u	displacement
A	area of cross section
A_{mn}	transmission coefficient
B	product of transmission coefficients
E	modulus of elasticity
$H(t-\tau)$	Heaviside function (0 if $t < \tau$ and 1 if $t > \tau$)
I	impulse (dimensions of pressure times time)
P	Laplace transform of p
U	Laplace transform of u
x_1, x_2, x_3	space coordinates along length of bar measured from left end of first, second, and third layers, respectively.

Abstract

The multi-layered cylinder made of elastic materials and subjected to an arbitrary end excitation is solved using the classical wave method, the Laplace transform method, and the transmissibility method. Complete numerical solutions are presented for a pressure pulse having a rapid rise and an exponential decay. Curves for displacements, stresses, and accelerations at various sections of bars of varying geometries are presented.

STRESS WAVE PROPAGATION
IN LAYERED ELASTIC CYLINDERS

1. Introduction. The problem of a semi-infinite solid under the action of a pressure pulse that is uniformly distributed over the surface at any instant of time is formally equivalent to the problem of a rod which suffers no lateral contraction and is acted upon by a time-varying pressure at its end. In this report three methods of solution for this problem are discussed; namely, the classical wave method, the Laplace transform method, and the transmissibility method.

Bars with one or more elastic layers are considered with the elastic properties varying from layer to layer. The geometries considered are shown in Figure 1.

(a)

(b)

(c)

(d)

(e)

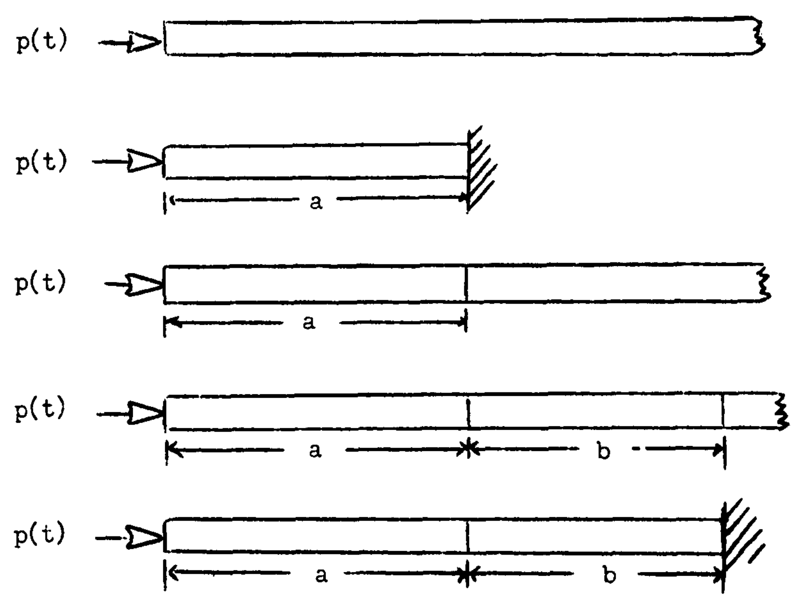


Figure 1

The solutions for the bars of Figure 1a and 1b are given first using the classical wave method. The Laplace transform method is used to find the solution for the bar of Figure 1c. This solution is shown to reduce to the solution found using the classical approach when $a = \infty$ (Figure 1a) and when the second layer is rigid (Figure 1b).

The transmissibility method is explained in conjunction with the rods of Figures 1a and 1b and is then used to solve the problem of Figure 1d. That solution is then compared with the Laplace transform solution and is also specialized to give the solution for Figure 1e.

The pressure pulses considered are the impulse of Figure 2a, the step function (or Heaviside function) of Figure 2b, and the exponentially decaying function of Figure 2c and 2d.

(a) $p = I\,\delta(t - \tau)$

$I = p_o\,\tau$

(b) $p = p_o\,H(t - \tau)$

(c) $p = \frac{p_o}{d-b}\left\{de^{-dt} - be^{-bt}\right\}$

(d) $p = p_o\left\{\frac{t}{t_r}\,G_o\,(t_r) + H(t - t_r)e^{-b(t-t_r)}\right\}$

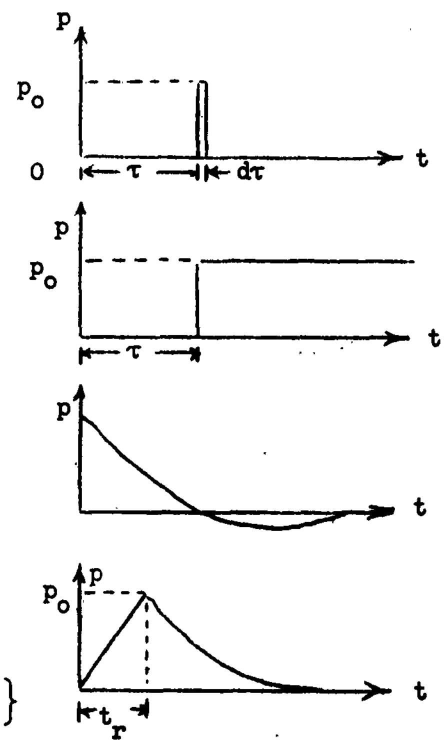


Figure 2

The first two are used to demonstrate the classical wave approach and to explain the transmissibility method. The pulse of Figure 2c is used in demonstrating the Laplace transform method. The pulse of Figure 2d, representing a practical shape which can be produced in the laboratory, is used to demonstrate the power of the transmissibility method. This report concludes with graphs representing displacement, acceleration and stress at various locations along the rod for the pulse of Figure 2d.

2. Basic Theory. The basic equation governing the response of a cylinder to a given end disturbance such as an applied pressure is based on the equation of motion which follows from Newton's second law and the stress-strain relation governing the behavior of the assumed material. For an elastic material, the normal stress σ is related to the normal strain ε through Hooke's law

$$\sigma = E\varepsilon \tag{1}$$

where E is Young's modules of elasticity. The normal strain is related to the axial displacement u through the relation

$$\varepsilon = \frac{\partial u}{\partial x} \tag{2}$$

where x is measured along the length of the bar. The equation of motion is easily derived if we consider a free body diagram of dx length of the bar as shown in Figure 3. The bar has the cross section area A and the mass density ρ . The normal stress assumed in the conventional manner to increase in the direction of increasing x. The normal force is given by the product of the uniformly distributed

normal stress and the area.

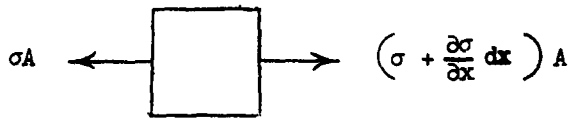


Figure 3

Applying Newton's second law of motion we find

$$\frac{\partial \sigma}{\partial x} \, dx \, A = \rho \, A \, \frac{\partial^2 u}{\partial t^2} \, dx \tag{3}$$

or

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} = \rho \, a \tag{4}$$

where a is the acceleration. This useful relation is valid for any continuous elastic body or to the material within a given layer of a layered elastic body.

Combining (1), (2), and (3) we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2} \tag{5}$$

or

$$u_{xx} = \frac{1}{c^2} \, u_{tt} \tag{6}$$

where the subscripts represent differentiation with respect to x or t. The constant c is the acoustic velocity for the material and is given by

$$c = \sqrt{\frac{E}{\rho}} \tag{7}$$

The partial differential equation (6) has the general solution

$$u = f\left(t - \frac{x}{c}\right) + g\left(t + \frac{x}{c}\right) \tag{8}$$

where f and g are arbitrary functions of $(t - \frac{x}{c})$ and $(t + \frac{x}{c})$, respectively. The first of these represents a progressive wave moving in the positive x direction while the second represents a regressive wave moving in the negative x direction. Associated with the displacement in (8) are the normal strain

$$\varepsilon = -\frac{1}{c} f'\left(t - \frac{x}{c}\right) + \frac{1}{c} g'\left(t + \frac{x}{c}\right) \tag{9}$$

the normal stress

$$\sigma = E\, u_x = -\frac{E}{c} f'\left(t - \frac{x}{c}\right) + \frac{E}{c} g'\left(t + \frac{x}{c}\right) \tag{10}$$

the velocity

$$v = u_t = f'\left(t - \frac{x}{c}\right) + g'\left(t + \frac{x}{c}\right) \tag{11}$$

and the acceleration

$$a = u_{tt} = f''\left(t - \frac{x}{c}\right) + g''\left(t + \frac{x}{c}\right) \tag{12}$$

where primes represent differentiation with respect to the argument $(t - \frac{x}{c})$ or $(t + \frac{x}{c})$, as the case may be. The waves move with the speed c.

3. Classical Wave Method. The classical approach to the problem is centered around a straight forward application of prescribed end conditions to the governing equation (8). Two elementary problems will demonstrate the method and will be useful later on as comparisons are made between the classical method and the other methods considered.

Two pressure pulses will be considered. The pulse of Figure 2a is applied at $x = o$ and is described mathematically by

$$\sigma(o,t) = - p_o \tau \delta(t) \tag{13}$$

where $\delta(t)$ is the Dirac delta function defined by

$$\delta(t) = \begin{cases} o & t < o \\ \infty & t = o \\ o & t > o \end{cases} \tag{14}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{15}$$

The product $p_o \tau$ is an impulse and may be denoted by the symbol I. The negative sign in (13) indicates a compressive stress. We may rewrite (13) as

$$\sigma(o,t) = - I \delta(t) \tag{16}$$

The pulse of Figure 2b when applied at $x = o$ is described mathematically by

$$\sigma(o,t) = - p_o H(t) \tag{17}$$

where $H(t)$ is the Heaviside function and is defined by

$$H(t) = \begin{cases} o & t < o \\ 1 & t > o \end{cases} \tag{18}$$

More generally, we may introduce the delay time t_o an[illegible] [illegible]ite

$$\delta(t - t_o) \equiv \delta_o(t) = \begin{cases} o & o < t < t_o \\ \infty & t = t_o \\ o & t > t_o \end{cases} \tag{19}$$

and

$$H(t - t_o) = H_o(t) = \begin{cases} o & o < t < t_o \\ 1 & t > t_o \end{cases} \tag{20}$$

Two bar geometries will be considered here: the semi-infinite bar of Figure 1a and the bar of finite length and right end fixed shown in Figure 1b.

(a) Semi-infinite Bar. Our first observation for the semi-infinite bar is that waves moving to the right are consistent with a pressure pulse applied at $x=o$ but waves moving to the left are not. We conclude, on physical grounds, that

$$g(t + \frac{x}{c}) = o \tag{21}$$

and thus

$$u = f(t - \frac{x}{c}) \tag{22}$$

The problem is to find the function $f(t - \frac{x}{c})$ which is consistent with the end condition associated with the applied pressure pulse.*
For the impulse of (16) we are led to try the general stress relation

$$\sigma(x,t) = - I\ \delta(t - \frac{x}{c}) \tag{23}$$

which, at least, satisfies (16). In terms of displacement, we use (10) to find

$$u_x = \mp \frac{I}{E}\ \delta(t - \frac{x}{c}) \tag{24}$$

or

$$u = + \frac{Ic}{E} H(t - \frac{x}{c}) \tag{25}$$

where

$$H(t - t_o) = \int_{-\infty}^{\infty} \delta(t - t_o)\, dt \tag{26}$$

*It is assumed that all bars treated are initially at rest.

Equation (25) is an acceptable solution for u since $f(t - \frac{x}{c})$ in (22) is of the correct form,

$$f(t - \frac{x}{c}) = \frac{Ic}{E} H(t - \frac{x}{c}) \qquad (27)$$

The impulse pressure thus gives rise to a step function type displacement. See Figure 4a.

For the step-function pressure we have (17). A likely expression for the stress at any section is

$$\sigma(x,t) = - p_o H(t - \frac{x}{c}) \qquad (28)$$

which leads to

$$u(x,t) = + \frac{p_o c}{E} (t - \frac{x}{c}) H(t - \frac{x}{c}) \qquad (29)$$

where

$$\int H(t - t_o) dt = (t - t_o) H(t - t_o) \qquad (30)$$

This is a ramp function indicating that the step pressure has associated with it a displacement which increases linearly from o at the wave front to an indefinitely large value behind the front. See Figure 4b.

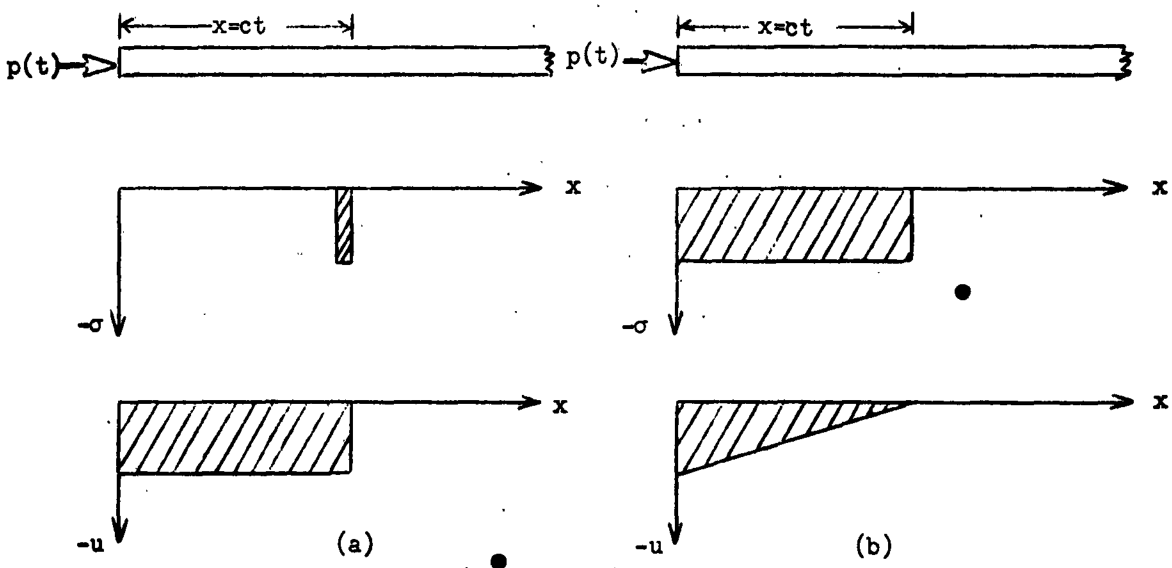


Figure 4

(b) Finite Bar with Fixed End. For the finite bar with one end ($x = a$) fixed we must find a displacement of the form (8) which satisfies the end conditions

$$\sigma(o, t) = - p(t) \tag{31}$$

and

$$u(a, t) = o \tag{32}$$

In this case waves will be reflected back from the fixed end and the functions $f(t - \frac{x}{c})$ and $g(t + \frac{x}{c})$ are both required. We must consider the solution appropriate to the time interval of interest. For example, when $t < a/c$, only the incident wave is present since reflection can not take place until an incident pulse reaches the fixed end. During this interval we have

$$\sigma(x,t) = - I\, \delta(t - \frac{x}{c}) \tag{33}$$

$$u(x,t) = + \frac{Ic}{E} H(t - \frac{x}{c}) \tag{34}$$

for the impulse. When $t < \frac{2a}{c}$, the end condition (32) assumes importance and we find

$$u(x,t) = \frac{Ic}{E} \left[H(t - \frac{x}{c}) - H(t - \frac{2a - x}{c}) \right] \tag{35}$$

When $t < \frac{3a}{c}$, there will be reflection from the left end and in order to satisfy both (31) and (32), we have

$$u(x,t) = \frac{Ic}{E} \left[H(t - \frac{x}{c}) - H(t - \frac{2a - x}{c}) - H(t - \frac{2a + x}{c}) \right] \tag{36}$$

For $t < \frac{4a}{c}$, we have

$$u(x,t) = \frac{Ic}{E}\left[H\left(t - \frac{x}{c}\right) - H\left(t - \frac{2a-x}{c}\right) - H\left(t - \frac{2a+x}{c}\right) + H\left(t - \frac{4a-x}{c}\right)\right] \tag{37}$$

Other terms may be added in order to extend the interval beyond $t = 4a/c$. For the step-function pressure pulse, (37) become

$$\begin{aligned} u(x,t) = \frac{p_o c}{E}\Big[&\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right) - \left(t - \frac{2a-x}{c}\right) H\left(t - \frac{2a-x}{c}\right) \\ &- \left(t - \frac{2a+x}{c}\right) H\left(t - \frac{2a+x}{c}\right) \\ &+ \left(t - \frac{4a-x}{c}\right) H\left(t - \frac{4a-x}{c}\right)\Big] \end{aligned} \tag{38}$$

Three important deductions should be made from the above development. (1) Reflected stresses are of the same sign as incident stresses when a fixed end is reached but reflected stresses are opposite in sign from incident stresses when a free end is reached; (2) displacements reflected from a fixed end are opposite in sign to the incident displacements but are of the same sign as the incident displacements when reflected from a free end; and (3) the solution for multiple reflections is tedious using the classical solution, requiring a skillful adjustment of wave functions in order to satisfy the given end conditions.

4. Laplace Transform Method. The basic element in the Laplace transform method is the transform operator itself. The function $f(t)$, for example, when multiplied by $e^{-st}\,dt$ and integrated from 0 to ∞ is defined as $F(s)$, the Laplace transform of $f(t)$. Thus,

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_0^{\infty} f(t)\,\bar{e}^{st}\,dt \qquad (39)$$

If the function to be transformed is dependent on two variables, say x and t, then

$$\mathcal{L}\left[f(x,t)\right] = F(x,s) = \int_0^{\infty} f(x,t)\,\bar{e}^{st}\,dt \qquad (40)$$

The transform of the first and second time derivatives of a function f(t) are found by performing an integration by parts as

$$\mathcal{L}\left[f_t\,(x,t)\right] = \int_0^{\infty} f_t\,(x,t)\,\bar{e}^{st}\,dt$$

$$= s\,F(x,s) - f(x,o) \qquad (41)$$

and

$$\mathcal{L}\left[\,f_{tt}\,(x,t)\,\right] = s^2\,F(x,s) - s\,f(x,o) - f_t(x,o) \qquad (42)$$

Having established these relations we now multiply each term (6) through by $\bar{e}^{st}$ dt and integrate from o to ∞ to obtain the transformed equation

$$U_{xx}(x,s) = \frac{1}{c^2}\left[s^2\,U(x,s) - s\,u(x,o) - u_t(x,o)\,\right] \qquad (43)$$

where

$$\mathcal{L}\left[u(x,t)\,\right] = U(x,s) \qquad (44)$$

If at t=o the bar is at rest, we have $u(x,o) = u_t(x,o) = o$ and (43) reduces to

$$U_{xx}(x,s) - \frac{s^2}{c^2}\,U(x,s) = o \qquad (45)$$

The partial differential equation in (x,t) space is thus reduced to an ordinary differential equation in (x,s) space.

At a fixed end (x = a), we have in (x,t) space

$$u(a,t) = o \tag{46}$$

In (x,s) space this becomes

$$U(a,s) = o \tag{47}$$

At a free end (x = o), we have in (x,t) space

$$\sigma(o,t) = Eu_x\,(o,t) = o \tag{48}$$

which in (x,s) space becomes

$$U_x(o,s) = o \tag{49}$$

At an end (x = o) with a prescribed pressure p(t), we have in (x,t) space

$$\sigma(o,t) = -\,p(t) = Eu_x(o,t) \tag{50}$$

and in (x,s) space

$$EU_x(o,s) = -\int_0^\infty p(t)\,\bar{e}^{\,st}\,dt = -P(s) \tag{51}$$

where $P(s)$ is the transform of $p(t)$. For the impulse pressure, step pressure, and the exponential pressure (Figure 2c) we have, respectively,

$$P(s) = I\int_0^\infty \delta(t)\,\bar{e}^{\,st}\,dt = I \tag{52}$$

$$P(s) = p_o\int_0^\infty H(t)\,\bar{e}^{\,st}\,dt = p_o/s \tag{53}$$

$$P(s) = \frac{p_o}{d-b}\int_0^\infty (d\bar{e}^{\,dt} - b\bar{e}^{\,bt})\,\bar{e}^{\,st}\,dt = \left(\frac{p_o}{d-b}\right)\left(\frac{d}{d+s} - \frac{b}{b+s}\right) \tag{54}$$

Now let us apply the Laplace transform method to the two layered bar of Figure 1c as redrawn in Figure 5.

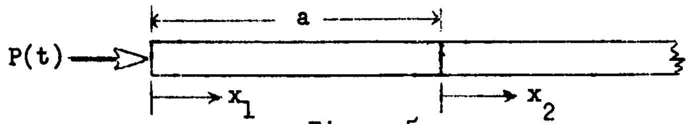


Figure 5

For this problem we have the acoustic velocities c_1 and c_2 for the two layers, the moduli of elasticity E_1 and E_2, the mass densities ρ_1 and ρ_2 with a length a for the finite portion. The prescribed end conditions are

$$\sigma(o,t) = -\, p(t) = Eu_x(o,t) \tag{55}$$

$$u_1(a,t) = u_2(o,t) \tag{56}$$

$$E_1 u_{1x}(a,t) = E_2\, u_{2x}(o,t) \tag{57}$$

$$u_2(\infty,\, t) = o \tag{58}$$

Equation (55) prescribes the end pressure, (56) prescribes that the displacements must be the same at the interface, (57) that the stresses must be the same at the interface and (58) that no disturbance be felt at the far end of the bar. The coordinates x_1 and x_2 are measured from the left ends of the respective layers.

The governing equations in (x,s) space are

$$U_{1xx}\,(x_1,s) - \frac{s^2}{c_1^2}\; U_1\,(x_1,s) = o \tag{59}$$

$$U_{2xx}\,(x_2,s) - \frac{s^2}{c_2^2}\; U_2\,(x_2,s) = o \tag{60}$$

Now let us apply the Laplace transform method to the two layered bar of Figure 1c as redrawn in Figure 5.

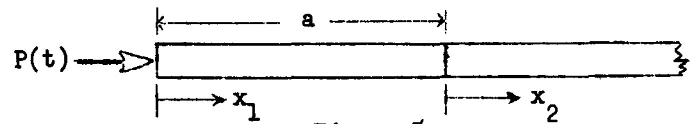


Figure 5

For this problem we have the acoustic velocities c_1 and c_2 for the two layers, the moduli of elasticity E_1 and E_2, the mass densities ρ_1 and ρ_2 with a length a for the finite portion. The prescribed end conditions are

$$\sigma(o,t) = -p(t) = Eu_x(o,t) \tag{55}$$

$$u_1(a,t) = u_2(o,t) \tag{56}$$

$$E_1 u_{1x}(a,t) = E_2 u_{2x}(o,t) \tag{57}$$

$$u_2(\infty, t) = o \tag{58}$$

Equation (55) prescribes the end pressure, (56) prescribes that the displacements must be the same at the interface, (57) that the stresses must be the same at the interface and (58) that no disturbance be felt at the far end of the bar. The coordinates x_1 and x_2 are measured from the left ends of the respective layers.

The governing equations in (x,s) space are

$$U_{1xx}(x_1,s) - \frac{s^2}{c_1^2} U_1(x_1,s) = o \tag{59}$$

$$U_{2xx}(x_2,s) - \frac{s^2}{c_2^2} U_2(x_2,s) = o \tag{60}$$

and the end condition are

$$U_{1x}\ (o,s) = -\frac{P(s)}{E_1} \qquad (61)$$

$$U_1\ (a,s) = U_2\ (o,s) \qquad (62)$$

$$E_1\ U_{1x}\ (a,s) = E_2\ U_{2x}\ (o,s) \qquad (63)$$

$$U_2\ (\infty,\ s) = o \qquad (64)$$

We solve (59) and (60) to obtain

$$U_1\ (x_1,s) = B_1\ e^{-sx_1/c_1} + B_2\ e^{sx_1/c_1} \qquad (65)$$

$$U_2\ (x_2,s) = B_3\ e^{-sx_2/c_2} + B_4\ e^{-sx_2/c_2} \qquad (66)$$

where B_1 , B_2, B_3, and B_4 are constants to be determined. Applying (64), we find $B_4 = o$. Applying (61), we find

$$B_1 - B_2 = \frac{c_1\ P(s)}{E_1 s} \qquad (67)$$

Applying (62) we get

$$B_3 = B_1\ e^{-sa/c_1} + B_2\ e^{sa/c_1} \qquad (68)$$

Applying (63) we get

$$B_3 = B_1 \left(\frac{E_1\ c_2}{E_2\ c_1}\right) e^{-sa/c_1} - B_2 \left(\frac{E_1\ c_2}{E_2\ c_1}\right) e^{sa/c_1} \qquad (69)$$

Solving these simultaneously, we introduce

$$A_{12} = \left[\frac{1 - \dfrac{E_1\, c_2}{E_2\, c_1}}{1 + \dfrac{E_1\, c_2}{E_2\, c_1}}\right] \qquad (70)$$

and find

$$B_1 = \frac{c_1\, P(s)}{E_1 s}\left[\frac{1}{1 + A_{12}\, e^{-2as/c_1}}\right] \qquad (71)$$

$$B_2 = -\frac{c_1\, P(s)}{E_1 s}\left[\frac{A_{12}\, e^{-2as/c_1}}{1 + A_{12}\, e^{-2as/c_1}}\right] \qquad (72)$$

$$B_3 = \frac{c_1\, P(s)}{E_1 s}\left[\frac{e^{-sa/c_1}\,(1 - A_{12})}{1 + A_{12}\, e^{-2sa/c_1}}\right] \qquad (73)$$

Now substitute into (65) and (66) to obtain

$$U_1\,(x_1, s) = \frac{c_1\, P(s)}{E_1 s(1 + A_{12}\, e^{-2as/c_1})}\left[e^{-sx_1/c_1} - A_{12}\, e^{-s(2a - x_1)/c_1}\right] \qquad (74)$$

$$U_2\,(x_2, s) = \frac{c_1\, P(s)\, e^{-sa/c_1}\,(1 - A_{12})\, e^{-sx_2/c_2}}{E_1 s\,(1 + A_{12}\, e^{-2sa/c_1})} \qquad (75)$$

Using the expansion

$$\frac{1}{1 + z} = \sum_{0}^{\infty} (-1)^n z^n \qquad (76)$$

we rewrite (74) and (75) in the form

$$U_1\,(x_1, s) = \frac{c_1\, P(s)}{E_1 s} \sum_{0}^{\infty} (-A_{12})^n \left[e^{-s(x_1 + 2an)/c_1} - A_{12} e^{-s(2a + 2an - x_1)/c_1}\right] \qquad (77)$$

$$U_2(x_2,s) = \frac{c_1\ P(s)\ (1-A_{12})\ e^{-sa/c_1}}{E_1 s} \sum_{o}^{\infty} (-A_{12})^n\ e^{-s(x_2\ +\ 2nac_2/c_1)/c_2} \tag{78}$$

For the special case of a semi-infinite bar of one layer $\rho_1 = \rho_2 = \rho$, $E_1 = E_2 = E$, and $c_1 = c_2 = c$, and we have $A_{12} = o$. Then (74) and (75) reduce directly to

$$U_1\ (x_1,s) = \frac{c\ P(s)\ e^{-sx_1/c}}{Es} \tag{79}$$

$$U_2\ (x_2,s) = \frac{c\ P(s)\ e^{-s(x_2+\ a)/c}}{Es} \tag{80}$$

These must be transformed finally from (x,s) space back to (x,t) space. This requires us to first state the exact nature of the pressure pulse whose Laplace transform is P(s). If, for example, we have the impulse pressure

$$p(t) = I\ \delta(t) \tag{81}$$

then from (52)

$$F(s) = I \tag{82}$$

and we have

$$U_1\ (x_1,s) = \frac{cI}{Es}\ e^{-sx_1/c} \tag{83}$$

$$U_2\ (x_2,s) = \frac{cI}{Es}\ e^{-s(x_2\ +\ a)/c} \tag{84}$$

It is easily shown that $(e^{-ks})/s$ is the transform of $H(t - k)$ and $U_1(x_1,s)$ and $U_2(x_2,s)$ are the transforms of $u_1(x_1,t)$ and $u_2(x_2,t)$. Thus,

$$u_1\ (x_1,t) = \frac{cI}{E}\ H(t - \frac{x_1}{c}) \tag{85}$$

$$u_2\ (x_2,t) = \frac{cI}{E}\ H(t - \frac{x_2 + a}{c}) \tag{86}$$

The second of these is quite clearly included in the first and we write simply

$$u(x,t) = \frac{cI}{E} H(t - \frac{x}{c}) \tag{87}$$

This checks the result found for the same problem in (25).

For the special case of the finite bar of length a and fixed right end we have $E_2 = \infty$ and $A_{12} = 1$. Equations (77) and (78) reduce in this case to

$$U_1\ (x_1,s) = \frac{c_1\ P(s)}{E_1 s} \sum_{0}^{\infty} (-1)^n \left[e^{-s(x_1 + 2an)/c_1} - e^{-s(2a + 2an - x_1)/c_1} \right] \tag{88}$$

$$U_2(x_2,s) = 0 \tag{89}$$

When the pressure is a step pulse we have

$$p(t) = p_o\ H(t) \tag{90}$$

and from (53), $P(s) = p_o/s$. Thus

$$U(x,s) = \frac{c\ p_o}{Es^2} \sum_{0}^{\infty} (-1)^n \left[e^{-s(x + 2an)/c} - e^{-s(2a + 2an - x)/c} \right] \tag{91}$$

where the subscript on x, E, and c have now been dropped. The transform of e^{-ks}/s^2 is $(t - k)\ H(t - k)$. Hence, we find the inverse transform of (91) to be

$$u(x,t) = \frac{c\ p_o}{E} \sum_{o}^{\infty} (-1)^n \left[(t - \frac{x + 2an}{c}) H(t - \frac{x + 2an}{c}) - (t - \frac{2a + 2an - x}{c}) H(t - \frac{2a + 2an - x}{c}) \right] \tag{92}$$

For the $n = o$ and $n = 1$ terms this expands to

$$u(x,t) = \frac{p_o c}{E} \left[(t - \frac{x}{c}) H(t - \frac{x}{c}) - (t - \frac{2a - x}{c}) H(t - \frac{2a - x}{c}) - (t - \frac{x + 2a}{c}) H(t - \frac{x + 2a}{c}) + (t - \frac{4a - x}{c}) H(t - \frac{4a - x}{c}) \right] \tag{93}$$

This is the complete solution if $t < {}^{4a}/c$. It checks (38) which was derived using the classical wave method.

Having established that the two layer solutions of (77) and (78) reduce to those found earlier for the single layer, we now convert the displacements in (x,s) space to (x,t) space. We consider particularly the exponential pressure pulse of Figure 20 having the transform (54). In (x,s) space, we have

$$U_1(x_1,s) = \frac{p_o\ c_1}{(d-b)E_1 s} \sum_{o}^{\infty} (-A_{12})^n \left[e^{-s(x + 2an)/c_1} - A_{12}\ e^{-s(2a + 2an - x_1)/c_1} \right] \left[\frac{d}{d+s} - \frac{b}{b+s} \right] \tag{94}$$

$$U_2(x_2,s) = \frac{p_o\ c_1\ (1 - A_{12})\ e^{-sa/c_1}}{(d - b)\ E_1 s} \sum_{o}^{\infty} (-A_{12})^n\ e^{-s(x_2 + 2nac_2/c_1)/c_2} \times \left[\frac{d}{d+s} - \frac{b}{b+s} \right] \tag{95}$$

At this point it is convenient to note a great simplification in finding the inverse transforms of expressions such as (94) and (95). In general, the inverse transform of $P(s)/s$ is given by

$$\mathcal{L}^{-1}\left[P(s)/s \right] = \int_0^t p(t)\, dt \tag{96}$$

We now write (94) and (95) in (x,t) space using

$$\mathcal{L}^{-1}\left[\frac{P(s)}{s} \right] = \int_0^t (de^{-dt} - be^{-bt})\, dt = -e^{-dt} + e^{-bt} \tag{97}$$

The exponential terms in (94) and (95) give rise to a translation of the multiplicative function. Thus

$$u_1(x_1,t) = \frac{p_o\, c_1}{E_1(d-b)} \sum_{0}^{\infty} (-A_{12})^n \Big[-e^{-d\left(t - \frac{x + 2an}{c_1}\right)} - A_{12}\, e^{-d\left(t - \frac{2a+2an-x_1}{c_1}\right)}$$
$$+ e^{-b\left(t - \frac{x + 2an}{c_1}\right)} - A_{12}\, e^{b\left(t - \frac{2a+2an-x_1}{c_1}\right)} \Big] \tag{98}$$

$$u_2(x_2,t) = \frac{p_o\, c_1}{E_1(d-b)} \sum_{0}^{\infty} (-A_{12})^n \Big[-e^{-d\left(t - \frac{x_2 + 2nac_2/c_1 + ac_2/c_1}{c_1}\right)}$$
$$+ e^{-b\left(t - \frac{x_2 + 2nac_2/c_1 + ac_2/c_1}{c_2}\right)} \Big] (1 - A_{12}) \tag{99}$$

where, for simplicity, the Heaviside functions, similar to those appearing in (92), have been omitted. Consequently, any term in (98) or (99) must be taken as zero whenever the parenthetical expression in the exponent for that term is negative.

5. Transmissibility Method. The transmissibility method discussed here is based on a direct physical interpretation of the results found in the previous section using the purely mathematical technique of the Laplace transform. The object is to be able to come to the same results in less time and to have a better appreciation of the physical behavior of the system. We begin by making several general observations.

First, in equations (77) and (78) we note that every term of the series on the right hand side of the equation is of the form

$$U(x,s) = \frac{c_1}{E_1} \frac{P(s)}{s} e^{-sk} B \tag{100}$$

where the constant B includes constants such as A_{12}, A_{23}, etc. While this is specifically shown for the bar of two layers, it is generally true for bars of any number of layers. An increasing number of layers only introduces more constants such as A_{12} into B. The quantity k in the exponent of (100) will be seen to depend on c, u, x, and the lengths of the layers.

Second, U(x,s) is the transform of u(x,t). Its general form, shown in (100), is especially convenient since we can easily show that

$$u(x,t) = \frac{c_1 B}{E_1} \int_0^t p(t)\, dt \qquad \text{delayed by } k \tag{101}$$

For example, if

$$p(t) = I\, \delta(t) \tag{102}$$

as in (81)

and

$$P(s) \equiv \int_0^\infty p(t)\, e^{-st}\, dt \tag{103}$$

then

$$u(x,t) = \frac{c_1}{E_1} BI\, H(t - k) \tag{104}$$

where $H(t - k)$ is delayed in time by an amount k as compared with $H(t)$.

Third, we observe that the presence of an interface separating two layers of a bar causes a constant such as A_{12} (associated with the interface separating the first and second layers) to be introduced. A bar having five layers would necessarily have associated with it A_{12}, A_{23}, A_{34}, and A_{45}. In general

$$A_{mn} = \left(\frac{1 - \dfrac{E_m c_n}{E_n c_m}}{1 + \dfrac{E_m c_n}{E_n c_m}} \right) \tag{105}$$

where

$$A_{mn} = -1 \qquad \text{if} \quad \frac{E_n}{c_n} = 0$$

$$-1 < A_{mn} < 0 \qquad \text{if} \quad \frac{E_m}{c_m} > \frac{E_n}{c_n}$$

$$A_{mn} = 0 \qquad \text{if} \quad \frac{E_m}{c_m} = \frac{E_n}{c_n}$$

$$0 < A_{mn} < 1 \qquad \text{if} \quad \frac{E_m}{c_m} < \frac{E_n}{c_n}$$

$$A_{mn} = +1 \qquad \text{if} \quad \frac{E_n}{c_n} = \infty$$

Fourth, since the displacement $u(x,t)$ is a function of $(t - \frac{x}{c})$ in the general case, it follows that normal stress $\sigma(x,t)$, being proportional to the x derivative of $u(x,t)$, will be opposite in sign to the displacement. Similarly, velocity and acceleration, being related to the time derivatives of $u(x,t)$ will have the same signs as the displacement. A positive change in displacement is thus associated with a negative change in normal stress and positive changes in velocity and acceleration. The A_{12} term in (74), for example, represents a negative change in $U_1(x_1,t)$ but will represent a positive change in $\sigma_1(x_1,t)$.

With these observations in mind, we now study the results for the semi-infinite bar of one layer and the finite bar of one layer with right end fixed with the goal of learning how to predict the terms in the series associated with more complicated geometries.

For the semi-infinite bar of one layer we have the results (82) and (84) which are the same and may be written in the general case as

$$U(x,s) = \frac{c\,P(s)}{Es}\, e^{\frac{-sx}{c}} \tag{106}$$

This gives for the impulsive pressure $p(t) = I\,\delta(t)$,

$$u(x,t) = \frac{cI}{E}\, H(t - \frac{x}{c}) \tag{107}$$

$$\sigma(x,t) = -I\,\delta(t - \frac{x}{c}) \tag{108}$$

and for the step pulse,

$$u(x,t) = \frac{c\,P_o}{E}\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right) \tag{109}$$

$$\sigma(x,t) = -P_o\, H\left(t - \frac{x}{c}\right) \tag{110}$$

In this case, we have only an incident wave advancing in the x direction.

For the single layer, fixed end, bar we have in the general case from (77) with $A_{12} = 1$,

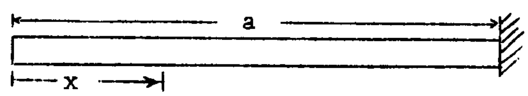
$$U(x,s) = \frac{c\,P(s)}{Es} \sum_{0}^{\infty} (-1)^n \left(e^{\frac{-s(x+2an)}{c}} - e^{\frac{-s(2a+2an-x)}{c}} \right) \tag{111}$$

Now expand this series to obtain

$$U(x,s) = \frac{c\,P(s)}{Es}\Bigg(e^{\frac{-sx}{c}} - e^{\frac{-s(2a-x)}{c}} - e^{\frac{-s(x+2a)}{c}} + e^{\frac{-s(4a-x)}{c}} + \\ + e^{\frac{-s(x+4a)}{c}} - e^{\frac{-s(6a-x)}{c}} - e^{\frac{-s(x+6a)}{c}} + e^{\frac{-s(8a-x)}{c}} + \\ + \ldots \Bigg) \tag{112}$$

The first term represents the incident wave [compare with (106)]. The second represents the wave which has traveled the length of the bar once and in addition has traveled the distance $a - x$. This is clearly the wave which has been reflected off the rigid and and is moving to the to the left. The third term represents the wave which has been reflected off the fixed end once and off the free end once and is moving toward the right.

Note that a sign change occurs in the displacement upon being bounced off the fixed end but no sign change occurs when bounced off the free end. A wave which has been bounced off the fixed end twice and the free end twice $(x + 4a)$ will have suffered two sign changes and if positive in the incident phase will still be positive. Normal stresses, on the other hand, suffers sign changes when bounced off a free end but not when bounced off a fixed end. Table I shows the values of B and k in (100) associated with each phase as related to a sketch indicating the distance traveled by the pulse.



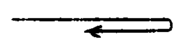
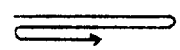
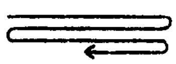
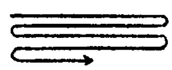
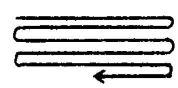
$k = x/c$	$k = (2a-x)/c$	$k = (x+2a)/c$
		
$B = +1$	$B = -1$	$B = -1$
$k = (4a-x)/c$	$k = (x+4a)/c$	$k = (6a-x)/c$
		
$B = +1$	$B = +1$	$B = -1$

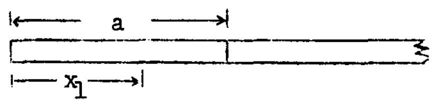
Table I

We now adapt this approach to the general problem of the two layered bar which is semi-infinite in length and governed by equations (77) and (78). The first term in the bracket of (77) represents

signals moving to the right in the first layer while the second term represents signals moving to the left in that layer. We expand (77) to get

$$U_1(x_1,s) = \frac{c_1 P(s)}{E_1 s}\left[e^{\frac{-sx_1}{c_1}} - A_{12}\, e^{\frac{-s(2a-x_1)}{c_1}} - A_{12}\, e^{\frac{-s(x_1+2a)}{c_1}} + A_{12}^2\, e^{\frac{-s(4a-x_1)}{c_1}} + A_{12}^2\, e^{\frac{-s(x_1+4a)}{c_1}} - A_{12}^3\, e^{\frac{-s(6a-x_1)}{c_1}} - A_{12}^3\, e^{\frac{-s(x_1+6a)}{c_1}} + A_{12}^4\, e^{\frac{-s(8a-x_1)}{c_1}} + \ldots \right] \quad (113)$$

We see again by comparing term 3 with 2, 5 with 4, and 7 with 6, that bouncing off the free end (the left end) multiplies the signal by a factor of +1. Bouncing off the interface, however, multiplies the signal by a factor of $-A_{12}$ (which is equal to -1 for a rigid interface). A signal which has bounced off the interface three times and off the free end twice is thus multiplied by a factor of $(-A_{12}^3)$. Table II shows the signal paths, delay times and coefficients for the first layer of the bar under discussion.



$k = x_1/c_1$ $B = 1$	$k = (2a-x_1)/c_1$ $B = A_{12}$	$k = (x_1+2a)/c_1$ $B = -A_{12}$
$k = (4a-x_1)/c_1$ $B = +A_{12}^2$	$k = (x_1+4a)/c_1$ $B = +A_{12}^2$	$k = (6a-x_1)/c_1$ $B = -A_{12}^3$

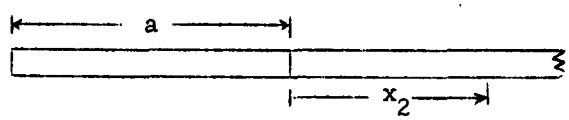
Table II

For the second layer, described by (78), we have

$$U_2(x_2,s) = \frac{c_1\ P(s)\ [1-A_{12}]}{E_1\ s}\Big(e^{-s(a/c_1+x_2/c_2)} - A_{12}\,e^{-s(3a/c_1+x_2/c_2)} + A_{12}^2\,e^{-s(5a/c_1+x_2/c_2)} - A_{12}^3\,e^{-s(7a/c_1+x_2/c_2)} + \dots\Big) \qquad (114)$$

where $x_2 + a = x_1$. Each of these terms represents a signal moving to

the right. The factor $[1 - A_{12}]$ represents the intensity factor for a signal which is carried across the interface from layer 1 to layer 2. When multiplied by the intensity of the signal in layer 1 which reaches the interface, we get the actual intensity of the signal transmitted into layer 2. The first term in (114) represents the signal which has come directly into layer 2 from the incident wave in layer 1. The second term represents the signal which has been bounced (within layer 1) off the interface once and off the free end once and is now transmitted through the interface into layer 2. Just as the factor $[1 - A_{12}]$ is necessary to indicate the diminution of intensity of the signal in passing through the interface, so also is a multiplicative factor of $-A_{12}$ necessary for each bounce off the interface before entering layer 2 and a factor of $(+1)$ for each bounce off the free end. Thus, a signal which bounced off the interface three times, off the free end twice, and then entered layer two would have a total factor of $-[1 - A_{12}]A_{12}^3$ as indicated by the fourth term of (114). Table III indicates the relevant information for the second layer.



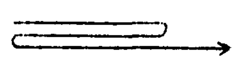
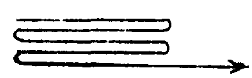
$k = a/c_1 + x_2/c_2$	$k = 3a/c_1 + x_2/c_2$	$k = 5a/c_1 + x_2/c_2$
		
$B = (1-A_{12})$	$B = -(1-A_{12})\,A_{12}$	$B = (1-A_{12})\,A_{12}^2$

Table III

6. Bar with Three Layers. We now use the physical understanding gained in the previous section as a means of writing with minimum formal mathematics the solution for a bar which is composed of three layers, the third of which extends to infinity on the right as shown in Figure 6.

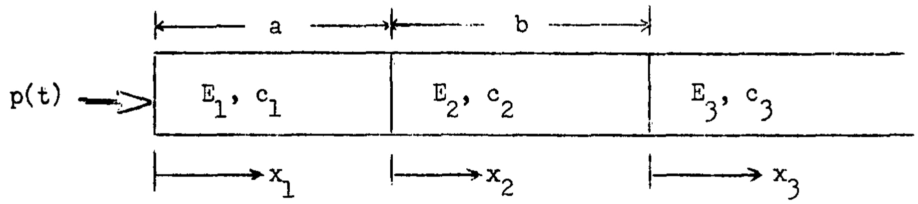


Figure 6

Associated with the interfaces we will have need for the constants A_{12} as defined by (70) and A_{23} defined by

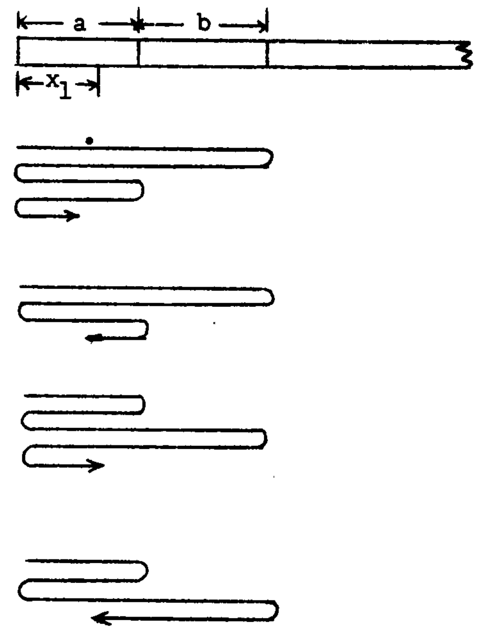
$$A_{23} = \left(\frac{1 - \dfrac{E_2\ c_3}{E_3\ c_2}}{1 + \dfrac{E_2\ c_3}{E_3\ c_2}}\right) \tag{115}$$

We now shall attempt to predict the possible paths followed by a signal before reaching a certain position in the bar and then state the constants B and k in (100) associated with the terms of an infinite series which correspond to those paths.

Consider first layer 1. A displacement signal entering this layer may bounce off the first interface or may be transmitted through the first interface and bounce off the second interface back into the second layer. Each time it bounces off the first interface it is multiplied by $-A_{12}$ if moving to the right, or by $+A_{12}$ if moving to the left. Each time it passes through the first interface moving to the right it is multiplied by $(1-A_{12})$. Each time it bounces off the second interface it is multiplied by $-A_{23}$. Each time it bounces off the free end it is multiplied by $(+1)$. Each time it passes through the first interface moving to the left it is multiplied by $(1+A_{12})$. We note further that every signal upon reaching a given point in the first layer will have traveled a total distance of $(2\alpha a+2\beta b+x_1)$ if moving to the right at the time in question where α and β are integers and a distance $(2\alpha a+2\beta b-x_1)$ if moving to the left where again α and β are positive integers or zero. If, for example, $\alpha=2$ and $\beta=1$ the path may be as shown in Figures 7a and 7b. The path associated with a given set (α,β) may not be unique as can be seen in Figure 7c and 7d. In general, we may say that the number of paths associated with a given set (α,β) is for the $+x_1$ and $-x_1$ terms,

$$N = \frac{(\alpha + \beta - 1)!}{\beta!\ (\alpha - 1)!} \tag{116}$$

For the example just considered, we find $N=2$, corresponding to the two cases shown in Figure 7.



(a) $4a + 2b + x_1$

(b) $4a + 2b - x_1$

(c) $4a + 2b + x_1$

(d) $4a + 2b - x_1$

Figure 7

The coefficient to be assigned to any term is dictated by the signal path and is given by the product of the factors associated with the penetration of each interface or the bouncing off each interface. For example, the path given in Figure 7b requires the coefficient in (100)

$$B = (1-A_{12})(-A_{23})(1+A_{12})(+1)(-A_{12}) = A_{12}\,(1-A_{12}^2)\,A_{23} \qquad (117)$$

The terms in parenthesis represent in order the penetration of the first interface, the bouncing off the second interface, the penetration of the first interface, the bouncing off the free end, and the bouncing

off the free end, and the bouncing off the first interface. It is evident that once the path is determined, the coefficient B is easily found. The delay time k in (100) is given by

$$k = \frac{4a+x_1}{c_1} + \frac{2b}{c_2} \tag{118}$$

Table IV shows the signal paths and B coefficients for all combinations of α and β from 0 to 3 for the first layer. The lines on the paths indicate the $(2\alpha a+2\beta b-x_1)$ cases while the dotted extension line represents the cases $(2\alpha a+2\beta b+x_1)$.

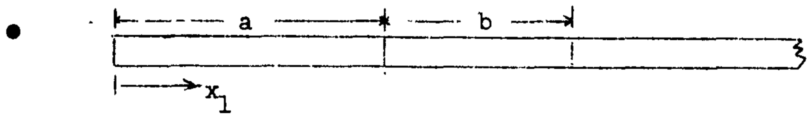
For the second layer, we follow the same general procedure in identifying the possible signal paths. The total path length will be of the form $\left[(2\alpha+1)a + 2\beta b \pm x_2\right]$, however for a given set of (α,β) values, the number of possible paths for signals moving to the right is given by

$$N = \frac{(\alpha + \beta)!}{\alpha!\ \beta!} \tag{119}$$

while the number of possible paths for signals moving to the left is given by

$$N = \frac{(\alpha + \beta - 1)!}{\alpha!(\beta - 1)!} \tag{120}$$

Table V shows the signal paths and B coefficients for values of $(2\alpha+1)$ from 0 to 25 and for 2β from 0 to 3 for signals moving to the right at a given time at a given section. Table VI shows similar values for signals moving to the left.



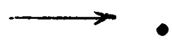
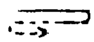
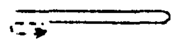
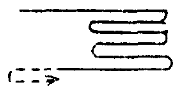
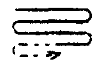
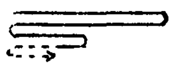
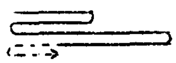
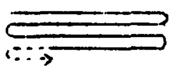
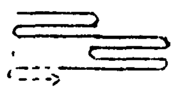
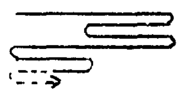
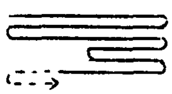
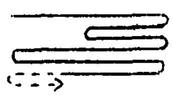
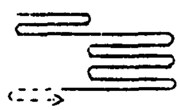
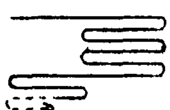
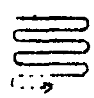
$\alpha = 0,\ \beta=0,\ N=1$  $B = 1$	$\alpha =1,\ \beta=0,\ N=1$  $B = A_{12}$	$\alpha =1,\ \beta=1,\ N=1$  $B = -(1-A_{12}^2)\ A_{23}$
$\alpha = 1,\ \beta=2,\ N=1$  $B = (1-A_{12}^2)\ A_{23}^2\ A_{12}$	$\alpha =1,\ \beta=3,\ N=1$  $B =-(1-A_{12}^2)\ A_{23}^3\ A_{12}^2$	$\alpha =2,\ \beta=0,\ N=1$  $B = A_{12}^2$
$\alpha = 2,\ \beta=1,\ N=2$  $B = (1-A_{12}^2)\ A_{23}A_{12}$	$\alpha =2,\ \beta=1,\ N=2$  $B = (1-A_{12}^2)\ A_{23}\ A_{12}$	
$\alpha =2,\ \beta=2,\ N=3$  $B = (1-A_{12}^2)^2\ A_{23}^2$	$\alpha =2,\ \beta=2,\ N=3$  $B =-(1-A_{12}^2)\ A_{23}^2\ A_{12}^2$	$\alpha =2,\ \beta=2,\ N=3$  $B =-(1-A_{12}^2)\ A_{23}^2\ A_{12}^2$
$\alpha =2,\ \beta=3,\ N=4$  $B = -(1-A_{12}^2)^2\ A_{23}^3\ A_{12}$	$\alpha =2,\ \beta=3,\ N=4$  $B = -(1-A_{12}^2)^2\ A_{23}^3\ A_{12}$	$\alpha =2,\ \beta=3,\ N=4$  $B = (1-A_{12}^2)\ A_{23}^3\ A_{12}^3$
$\alpha =2,\ \beta=3,\ N=4$  $B = (1-A_{12}^2)\ A_{23}^3\ A_{12}^3$		$\alpha =3,\ \beta=0,\ N=1$  $B = A_{12}^3$

Table IV

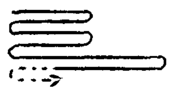
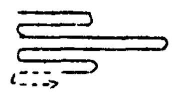
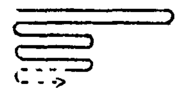
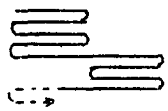
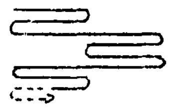
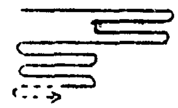
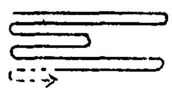
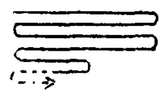
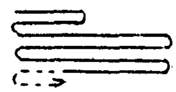
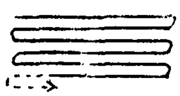
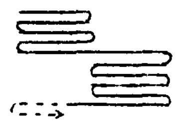
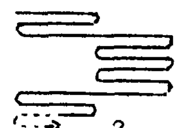
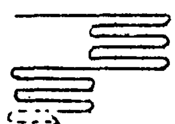
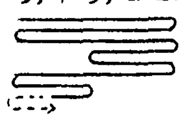
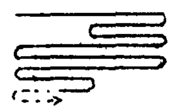
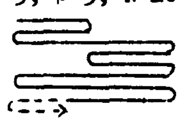
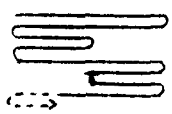
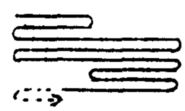
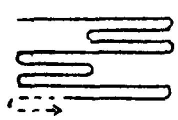
$\alpha=3,\ \beta=1,\ N=3$  $B = -(1-A_{12}^2)\ A_{23}\ A_{12}^2$	$\alpha=3,\ \beta=1,\ N=3$  $B = -(1-A_{12}^2)\ A_{23}\ A_{12}^2$	$\alpha=3,\ \beta=1,\ N=3$  $B = -(1-A_{12}^2)\ A_{12}^2\ A_{23}$
$\alpha=3,\ \beta=2,\ N=6$  $B = (1-A_{12}^2)\ A_{23}^2\ A_{12}^3$	$\alpha=3,\ \beta=2,\ N=6$  $B = (1-A_{12}^2)\ A_{23}^2\ A_{12}^3$	$\alpha=3,\ \beta=2,\ N=6$  $B = (1-A_{12}^2)\ A_{23}^2\ A_{12}^3$
$\alpha=3,\ \beta=2,\ N=6$  $B = -(1-A_{12}^2)^2 A_{23}^2\ A_{12}$	$\alpha=3,\ \beta=2,\ N=6$  $B = -(1-A_{12}^2)^2 A_{23}^2\ A_{12}$	$\alpha=3,\ \beta=2,\ N=6$  $B = -(1-A_{12}^2)^2\ A_{23}^2\ A_{12}$
$\alpha=3,\ \beta=3,\ N=10$  $B = -(1-A_{12}^2)^3\ A_{23}^3$	$\alpha=3,\ \beta=3,\ N=10$  $B = -(1-A_{12}^2)\ A_{23}^3\ A_{12}^4$	$\alpha=3,\ \beta=3,\ N=10$  $B = -(1-A_{12}^2)\ A_{23}^3\ A_{12}^4$
$\alpha=3,\ \beta=3,\ N=10$  $B = -(1-A_{12}^2)\ A_{23}^3\ A_{12}^4$	$\alpha=3,\ \beta=3,\ N=10$  $B = (1-A_{12}^2)^2\ A_{23}^3\ A_{12}^2$	$\alpha=3,\ \beta=3,\ N=10$  $B = (1-A_{12}^2)^2\ A_{23}^3\ A_{12}^2$
$\alpha=3,\ \beta=3,\ N=10$  $B = (1-A_{12}^2)^2\ A_{23}^3\ A_{12}^2$	$\alpha=3,\ \beta=3,\ N=10$  $B = (1-A_{12}^2)^2\ A_{23}^3\ A_{12}^2$	$\alpha=3,\ \beta=3,\ N=10$  $B = (1-A_{12}^2)^2\ A_{23}^3\ A_{12}^2$
$\alpha=3,\ \beta=3,\ N=10$  $B = (1-A_{12}^2)^2\ A_{23}^3\ A_{12}^2$		

Table IV (Continued)

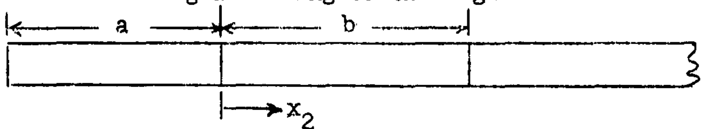
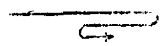
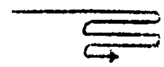
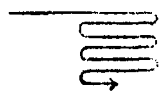
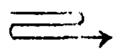
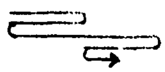
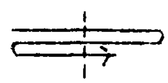
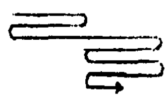
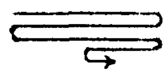
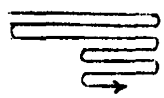
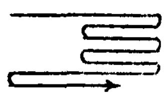
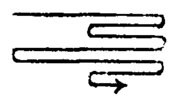
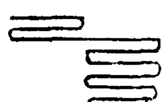
Signal moving to the right 		
$\alpha=0,\ \beta=0,\ N=1$ $B = (1-A_{12})$	$\alpha=0,\ \beta=1,\ N=1$  $B = -(1-A_{12})\ A_{23}\ A_{12}$	$\alpha=0,\ \beta=2,\ N=1$  $B = (1-A_{12})\ A_{23}^{2}\ A_{12}^{2}$
$\alpha=0,\ \beta=3,\ N=1$  $B = -(1-A_{12})\ A_{23}^{3}\ A_{12}^{3}$	$\alpha=1,\ \beta=0,\ N=1$  $B = -(1-A_{12})\ A_{12}$	
$\alpha=1,\ \beta=1,\ N=2$  $B = (1-A_{12})\ A_{23}\ A_{12}^{2}$	$\alpha=1,\ \beta=1,\ N=2$  $B = -(1-A_{12}^{2})(1-A_{12})\ A_{23}$	
$\alpha=1,\ \beta=2,\ N=3$  $B = -(1-A_{12})\ A_{23}^{2}\ A_{12}^{3}$	$\alpha=1,\ \beta=2,\ N=3$  $B = (1-A_{12}^{2})(1-A_{12})A_{23}^{2}A_{12}$	$\alpha=1,\ \beta=2,\ N=3$  $B = (1-A_{12}^{2})(1-A_{12})A_{23}^{2}A_{12}$
$\alpha=1,\ \beta=3,\ N=4$  $B=-(1-A_{12}^{2})(1-A_{12})A_{23}^{3}A_{12}^{2}$	$\alpha=1,\ \beta=3,\ N=4$  $B = -(1-A_{12}^{2})(1-A_{12})A_{23}^{3}A_{12}^{2}$	$\alpha=1,\ \beta=3,\ N=4$  $B = -(1-A_{12})(1-A_{12})A_{23}^{3}A_{12}^{2}$
$\alpha=1,\ \beta=3,\ N=4$  $B = (1-A_{12})\ A_{23}^{3}\ A_{12}^{4}$		$\alpha=2,\ \beta=0,\ N=1$  $B = -(1-A_{12})\ A_{12}$

Table V

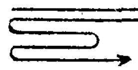
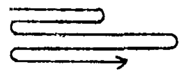
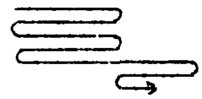
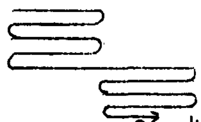
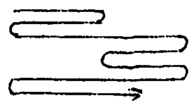
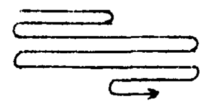
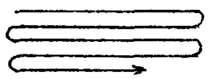
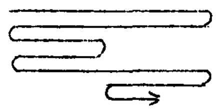
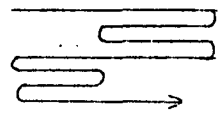
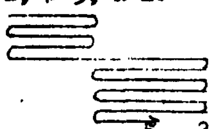
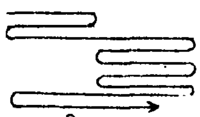
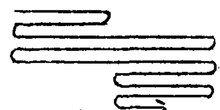
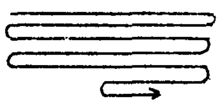
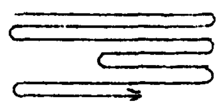
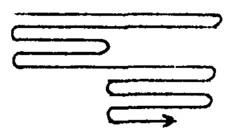
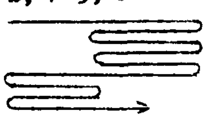
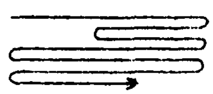
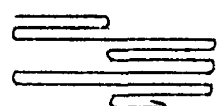
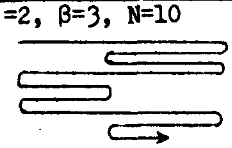
$\alpha=2,\ \beta=1,\ N=3$  $B=(1-A_{12}^2)(1-A_{12})A_{23}A_{12}$	$\alpha=2,\ \beta=1,\ N=3$  $B=(1-A_{12}^2)(1-A_{12})A_{23}A_{12}$	$\alpha=2,\ \beta=1,\ N=3$  $B=-(1-A_{12})\ A_{23}\ A_{12}^3$
$\alpha=2,\ \beta=2,\ N=6$  $B=(1-A_{12})\ A_{23}^2\ A_{12}^4$	$\alpha=2,\ \beta=2,\ N=6$  $B=-(1-A_{12}^2)(1-A_{12})A_{23}^2A_{12}^2$	$\alpha=2,\ \beta=2,\ N=6$  $B=-(1-A_{12}^2)(1-A_{12})A_{23}^2A_{12}^2$
$\alpha=2,\ \beta=2,\ N=6$  $B=(1-A_{12}^2)^2(1-A_{12})\ A_{23}^2$	$\alpha=2,\ \beta=2,\ N=6$  $B=-(1-A_{12}^2)(1-A_{12})A_{23}^2A_{12}^2$	$\alpha=2,\ \beta=2,\ N=6$  $B=-(1-A_{12}^2)(1-A_{12})A_{23}^2A_{12}^2$
$\alpha=2,\ \beta=3,\ N=10$  $B=-(1-A_{12})\ A_{12}^5\ A_{23}^3$	$\alpha=2,\ \beta=3,\ N=10$  $B=(1-A_{12}^2)(1-A_{12})A_{23}^3A_{12}^3$	$\alpha=2,\ \beta=3,\ N=10$  $B=(1-A_{12}^2)(1-A_{12})A_{23}^3A_{12}^3$
$\alpha=2,\ \beta=3,\ N=10$  $B=-(1-A_{12}^2)^2(1-A_{12})A_{23}^3A_{12}$	$\alpha=2,\ \beta=3,\ N=10$  $B=-(1-A_{12}^2)^2(1-A_{12})A_{23}^3A_{12}$	$\alpha=2,\ \beta=3,\ N=10$  $B=(1-A_{12}^2)(1-A_{12})A_{23}^3A_{12}^3$
$\alpha=2,\ \beta=3,\ N=10$  $B=(1-A_{12}^2)(1-A_{12})A_{23}^3A_{12}^3$	$\alpha=2,\ \beta=3,\ N=10$  $B=-(1-A_{12}^2)^2(1-A_{12})A_{23}^3A_{12}$	$\alpha=2,\ \beta=3,\ N=10$  $B=(1-A_{12}^2)(1-A_{12})A_{23}^3A_{12}^3$
$\alpha=2,\ \beta=3,\ N=10$  $B=(1-A_{12}^2)(1-A_{12})A_{23}^3A_{12}^3$		

Table V (Continued)

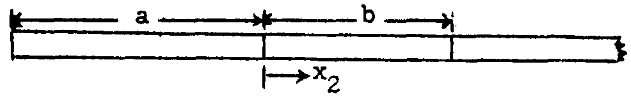
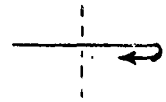
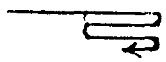
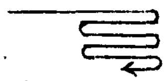
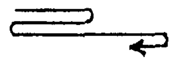
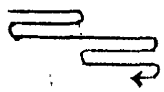
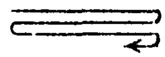
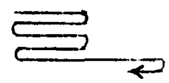
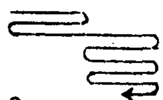
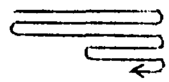
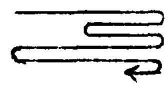
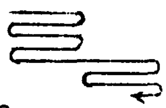
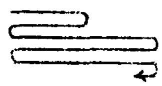
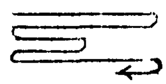
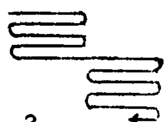
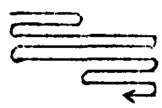
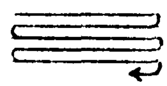
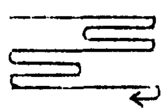
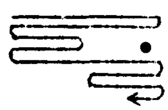
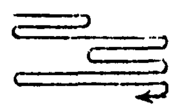
 Signals moving to the left		$\alpha=0,\ \beta=1,\ N=1$  $B=-(1-A_{12})\,A_{23}$
$\alpha=0,\ \beta=2,\ N=1$  $B=+A_{12}A_{23}^2(1-A_{12})$	$\alpha=0,\ \beta=3,\ N=1$  $B=-A_{12}^2A_{23}^3(1-A_{12})$	$\alpha=1,\ \beta=1,\ N=1$  $B=+A_{12}A_{23}(1-A_{12})$
$\alpha=1,\ \beta=2,\ N=2$  $B=-A_{12}^2A_{23}^2(1-A_{12})$	$\alpha=1,\ \beta=2,\ N=2$  $B=A_{23}^2(1-A_{12})(1-A_{12}^2)$	$\alpha=2,\ \beta=1,\ N=1$  $B=-A_{12}^2A_{23}(1-A_{12})$
$\alpha=1,\ \beta=3,\ N=3$  $B=+A_{12}^3A_{23}^3(1-A_{12})$	$\alpha=1,\ \beta=3,\ N=3$  $B=-A_{12}A_{23}^3(1-A_{12})(1-A_{12}^2)$	$\alpha=1,\ \beta=3,\ N=3$  $B=-A_{12}A_{23}^3(1-A_{12})(1-A_{12}^2)$
$\alpha=2,\ \beta=2,\ N=3$  $B=A_{12}^3A_{23}^2(1-A_{12})$	$\alpha=2,\ \beta=2,\ N=3$  $B=-A_{12}A_{23}^2(1-A_{12})(1-A_{12}^2)$	$\alpha=2,\ \beta=2,\ N=3$  $B=-A_{12}A_{23}^2(1-A_{12})(1-A_{12}^2)$
$\alpha=2,\ \beta=3,\ N=6$  $B=-A_{12}^4A_{23}^3\ (1-A_{12})$	$\alpha=2,\ \beta=3,\ N=6$  $B=A_{12}^2A_{23}^3(1-A_{12})(1-A_{12}^2)$	$\alpha=2,\ \beta=3,\ N=6$  $B=-A_{23}^3(1-A_{12}^2)^2(1-A_{12})$
$\alpha=2,\ \beta=3,\ N=6$  $B=A_{12}^2A_{23}^3(1-A_{12})(1-A_{12}^2)$	$\alpha=2,\ \beta=3,\ N=6$  $B=A_{12}^2A_{23}^3(1-A_{12})(1-A_{12}^2)$	$\alpha=2,\ \beta=3,\ N=6$  $B=A_{12}^2A_{23}^3(1-A_{12})(1-A_{12}^2)$

Table VI

Values of k for each signal path are found using the expression

$$k = \frac{(2\alpha+1)a}{c_1} + \frac{2\beta b}{c_2} \pm \frac{x_2}{c_2} \tag{121}$$

Each signal path with its associated values of B and k may be identified with a term such as (100). All such terms added together will yield the series which represents the displacement at any point in the layer considered expressed in terms of (x,s) space. Equation (113), it will be recalled, was such an expression for the first layer of the two layered bar.

We translate the series in (x,s) space into a series for (x,t) space by performing the inverse Laplace transform as explained in the previous section. The general term (100) will transform, for example, into the term (101). For the impulse $I\,\delta(T)$ this becomes (104).

The normal stress is found by multiplying the first x derivative of $u(x,t)$ by the value of E in the layer. The acceleration is found by taking the second t derivative of $u(x,t)$.

6. <u>Numerical Example.</u> As a numerical example of practical importance, we consider a pressure pulse as depicted in Figure 8. It was chosen to simulate a pressure - time function actually reproducible

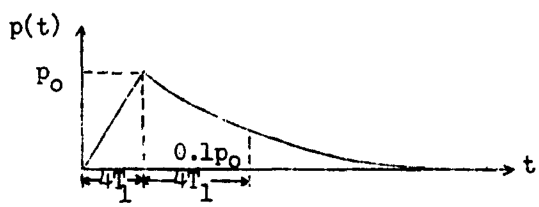


Figure 8

in the laboratory and has the equation

$$p(t) = p_o \left\{ \frac{t}{4T_1} G_o(4T_1) + H(t-4T_1)\, e^{-0.575(t/T_1-4)} \right\} \tag{122}$$

where T_1, the travel time for a wave moving through the first layer is given by

$$T_1 = a/c_1 \tag{123}$$

and

$$G_o(4T_1) \equiv H(t) - H(t-4T_1) \tag{124}$$

The bars considered have the geometric properties shown in Figure 9. The length b of the second layer is chosen so that the travel time T_2 of a wave through that layer will be exactly 1.5 times the travel time of a wave through the first layer. Thus

$$b = 1.5ac_2/c_1 \tag{125}$$

Displacements and stresses as functions of time were determined at locations $x_1=a/3$, $x_1=a$, $x_2=b/3$ and $x_2=b$. Accelerations were determined at $x_1 = a/3$ and $x_2 = b/3$.

The physical properties of the materials in the bar are assumed such that $E_1/c_1 = 2E_2/3c_2$ and $3E_3/c_3 = 7E_2/c_2$.

To fix ideas, we now choose a particular signal path associated with a particular point in the bar and find the related term in the series expansion for $u(x,t)$. We choose as an example the point $x_1 = a/3$ in the bar of Figure 9c and the path associated with $\alpha = 2$ and $\beta = 1$ and a signal moving to the left. This gives a value of k in (100) of

$$k = (4a-x_1)/c_1 + 2b/c_2 \tag{126}$$

(a)

(b)

(c)

(d)

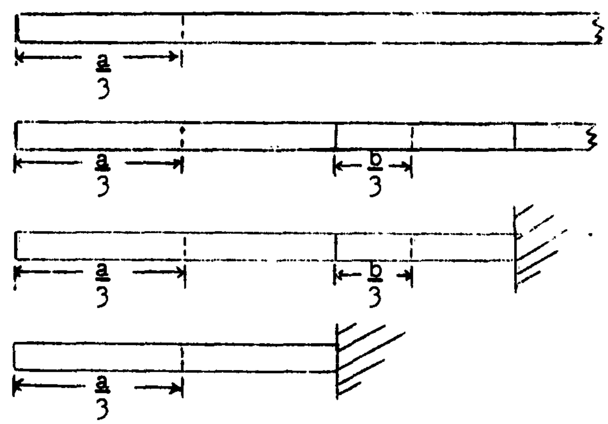


Figure 9

The transmission coefficients in this case are found using (105) to be

$$A_{12} = 0.2 \qquad A_{23} = 1.0 \tag{127}$$

From Table VI we now find B to be

$$B = (1-0.04)(1.0)(0.2) \equiv 0.192 \tag{128}$$

The associated term in (x,s) space is

$$U(x_1,s) = -\frac{0.192\, c_1\, P(s)}{E_1 s}\, e^{-s(4a/c_1+2b/c_2-x_1/c_1)} \tag{129}$$

To facilitate matters, we now use (96) to find the inverse transform of $P(s)/s$ where $p(t)$ is given by (122). This yields

$$\mathcal{L}^{-1}\left[P(s)/s\right] = p_o\left\{\frac{t^2}{8T_1}\, G_o(4T_1) + \frac{T_1\, H(t-4T_1)}{0.575}\left[1-e^{-0.575(t-4T_1)/T_1}\right]\right\} \tag{130}$$

The inverse transform of (129) is then

$$u(x_1,t) = -\frac{0.192\ c_1\ p_o}{E_1}\left\{\frac{(t-k)^2}{8T_1}\ G_k(4T_1) + \right.$$

$$\left. + \frac{T_1\ H(t-4T_1-k)}{0.575}\left[1-e^{-0.575(t-4T_1-k)/T_1}\right]\right\} \quad (131)$$

where

$$G_k(4T_1) \equiv H(t-k) - H(t-4T_1-k) \quad (132)$$

We now proceed to plot this curve. While the expression (131) may appear to be tedious to plot it will be found with a little practice that it offers no great difficulty. In a like manner, we calculate every other term in the expansion for $u(x,t)$ and plot it. The sum of all terms is easily found graphically by summing the ordinates of all the curves.

The displacement at $x_1=a/3$ for the bar in Figure 9c is shown in Figure 10. The curves for several of the typical terms in the series expansion are shown as is the curve representing the sum. Because every term was not considered in arriving at the sum, the resultant curve is not precise quantitatively although it clearly describes the correct trend. Figures 11, 12, 13, and 14 show the resultant displacement curves for all the bars of Figure 9 at $x_1=a/3$, $x_1=a$, $x_2=b/3$ and $x_2=b$.

The stress may be found directly from the displacement equation (131) to be

$$\sigma(x_1,t) = -\ 0.192\ p_o \left\{ \frac{(t-k)}{4T_1} G_k(4T_1) + \right.$$

$$\left. + H(t-4T_1-k)\ e^{-0.575(t-4T_1-k)/T_1} \right\} \tag{133}$$

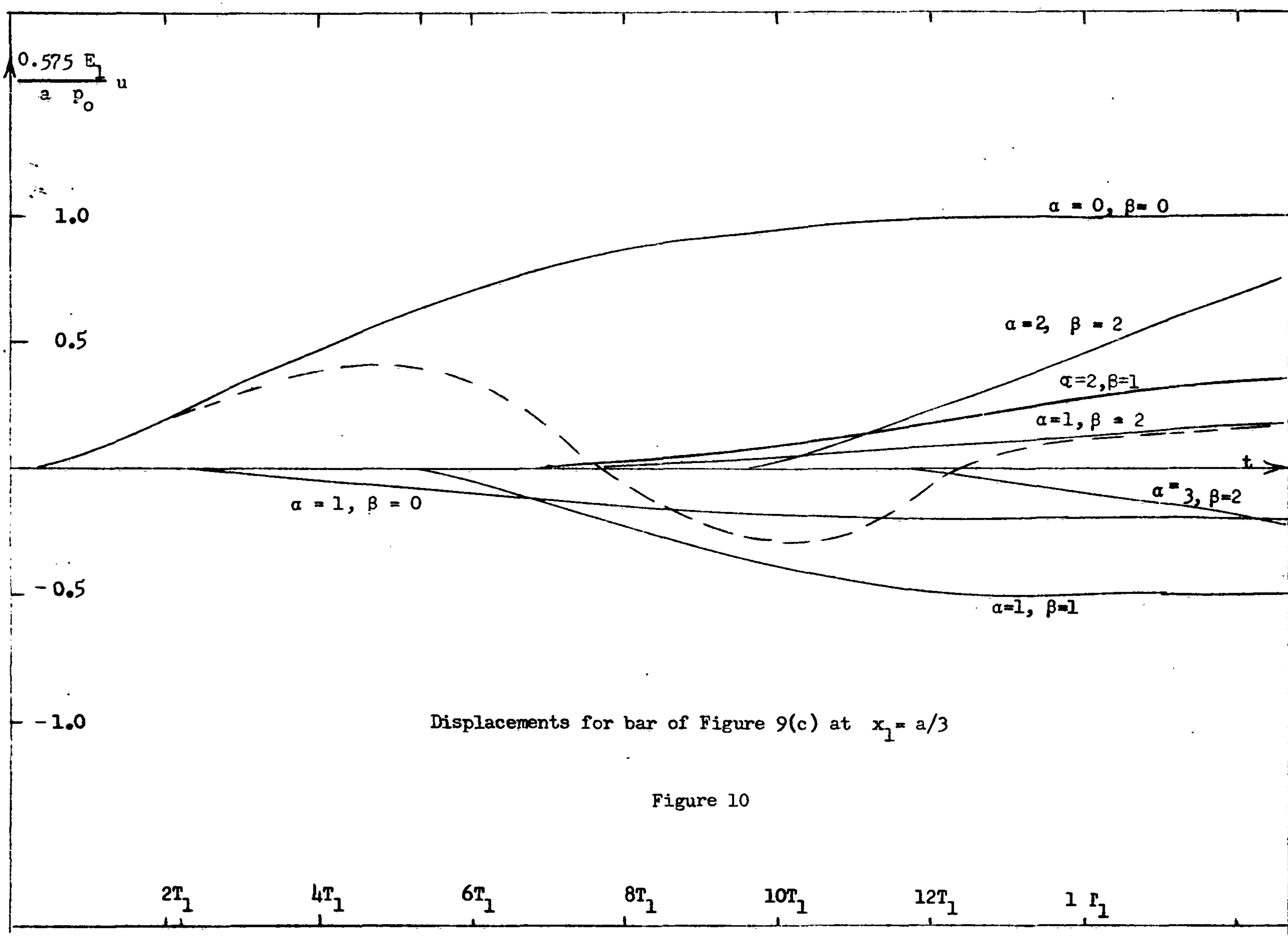
Curves for this and several other terms of the series are plotted in Figure 15. Figures 16, 17, 18, and 19 show the resultant stress curves for all bars of Figure 9 at $x_1=a/3$, $x_1=a$, $x_2=b/3$, and $x_2=b$.

The acceleration is also found from the displacement expression (131). It is

$$u_{tt}(x_1,t) = -\frac{0.192\ c_1\ p_o}{E_1} \left\{ \frac{1}{4T_1} G_k(4T_1) - \right.$$

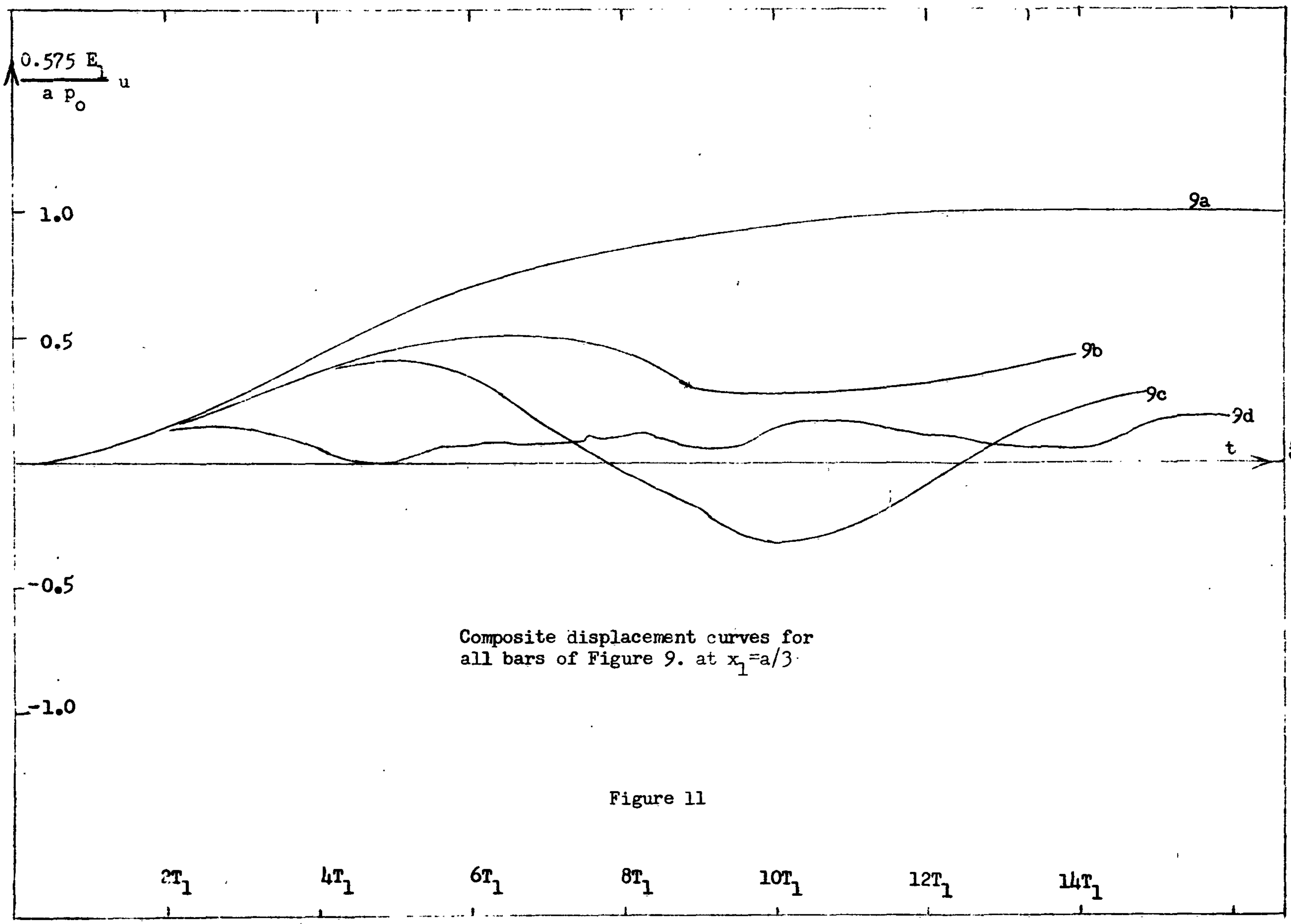
$$\left. -\ 0.575\ H(t-4T_1-k)\ e^{-0.575(t-4T_1-k)/T_1} \right\} \tag{134}$$

This equation as well as similar equations for several other terms appearing in the series expansion for acceleration are plotted in Figure 20. In Figures 21 and 22 resultant acceleration curves are plotted for all the bars of Figure 9 at $x_1=a/3$ and $x_2=b/3$.



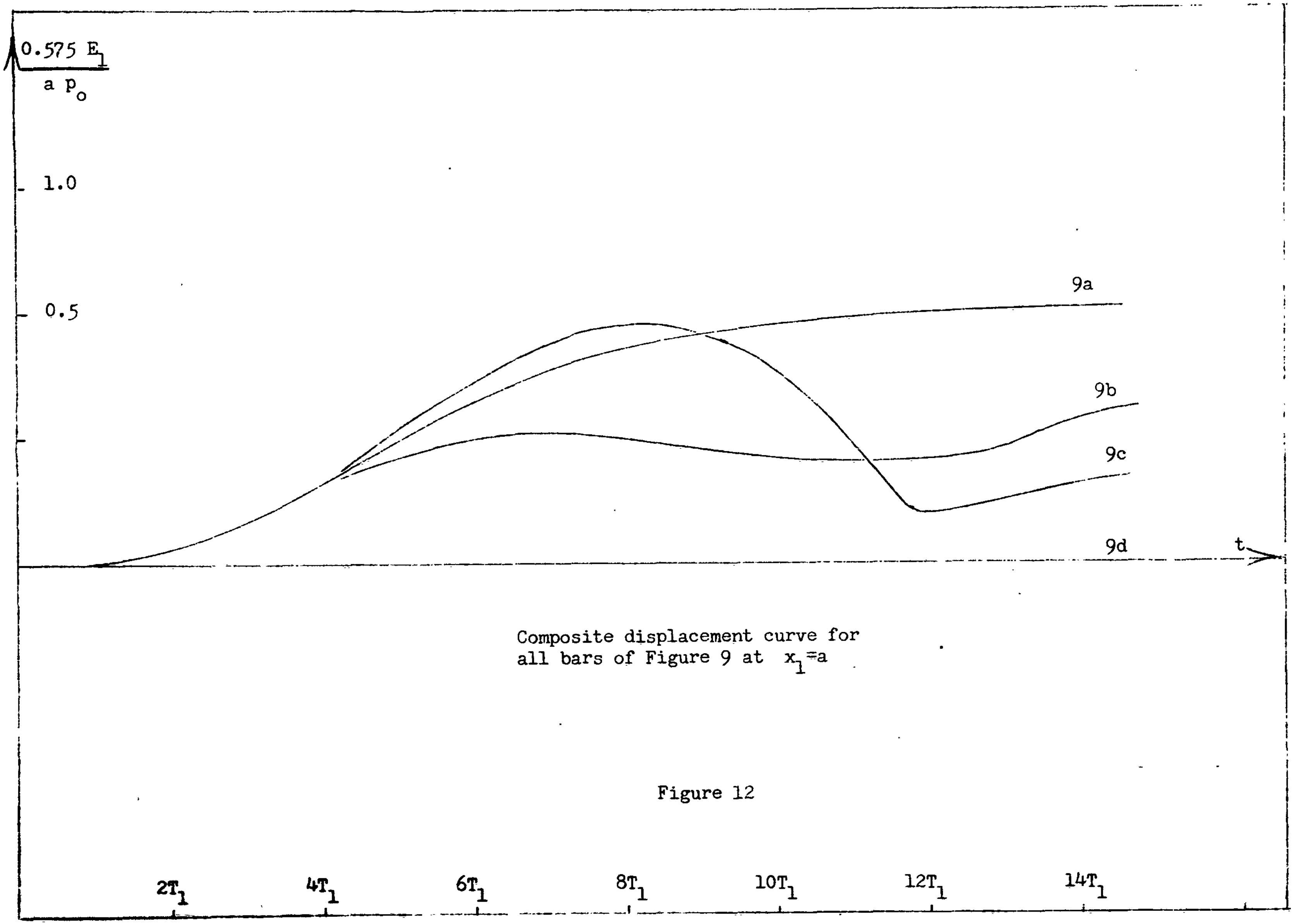
Displacements for bar of Figure 9(c) at $x_1 = a/3$

Figure 10



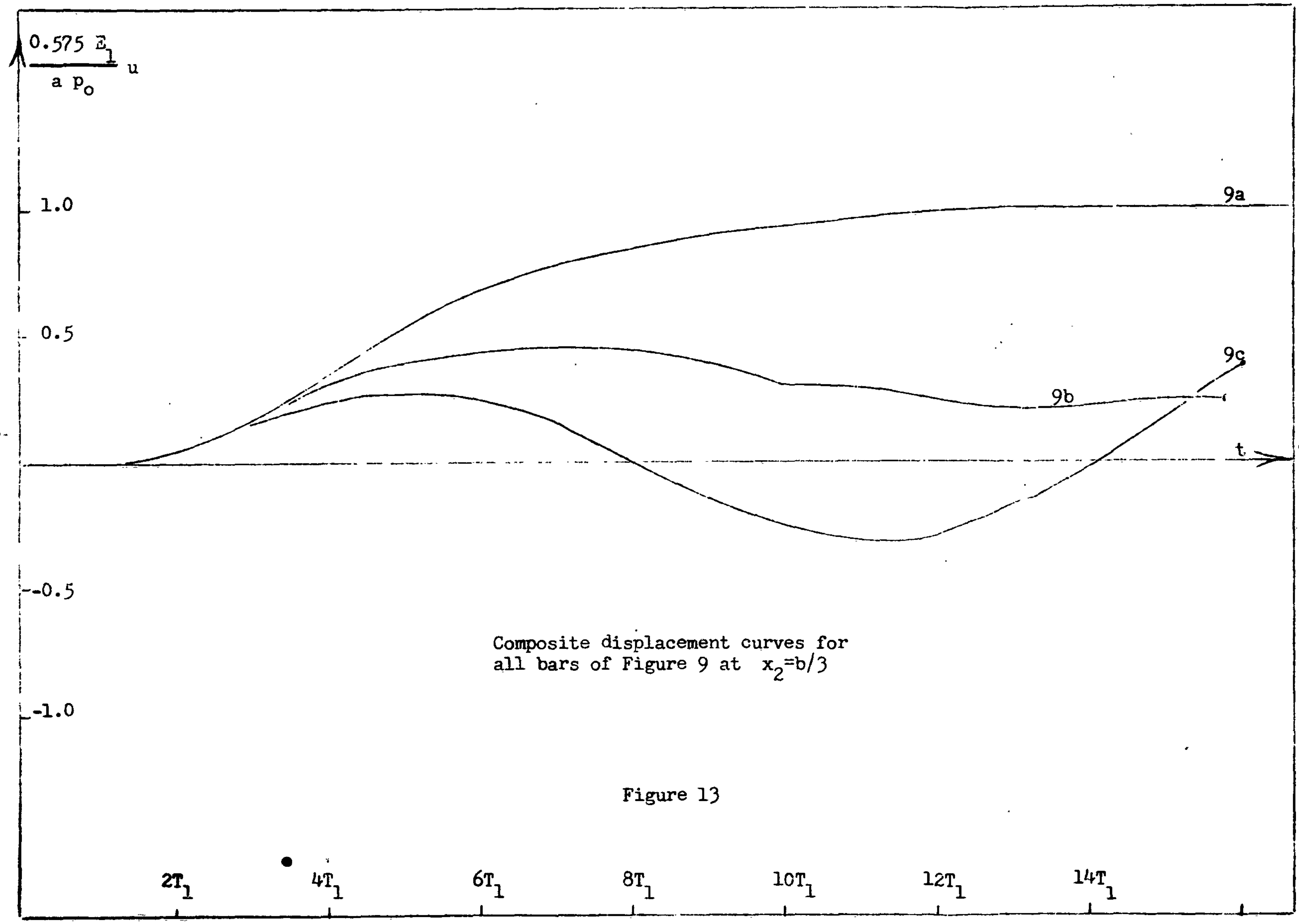
Composite displacement curves for all bars of Figure 9. at x_1=a/3

Figure 11



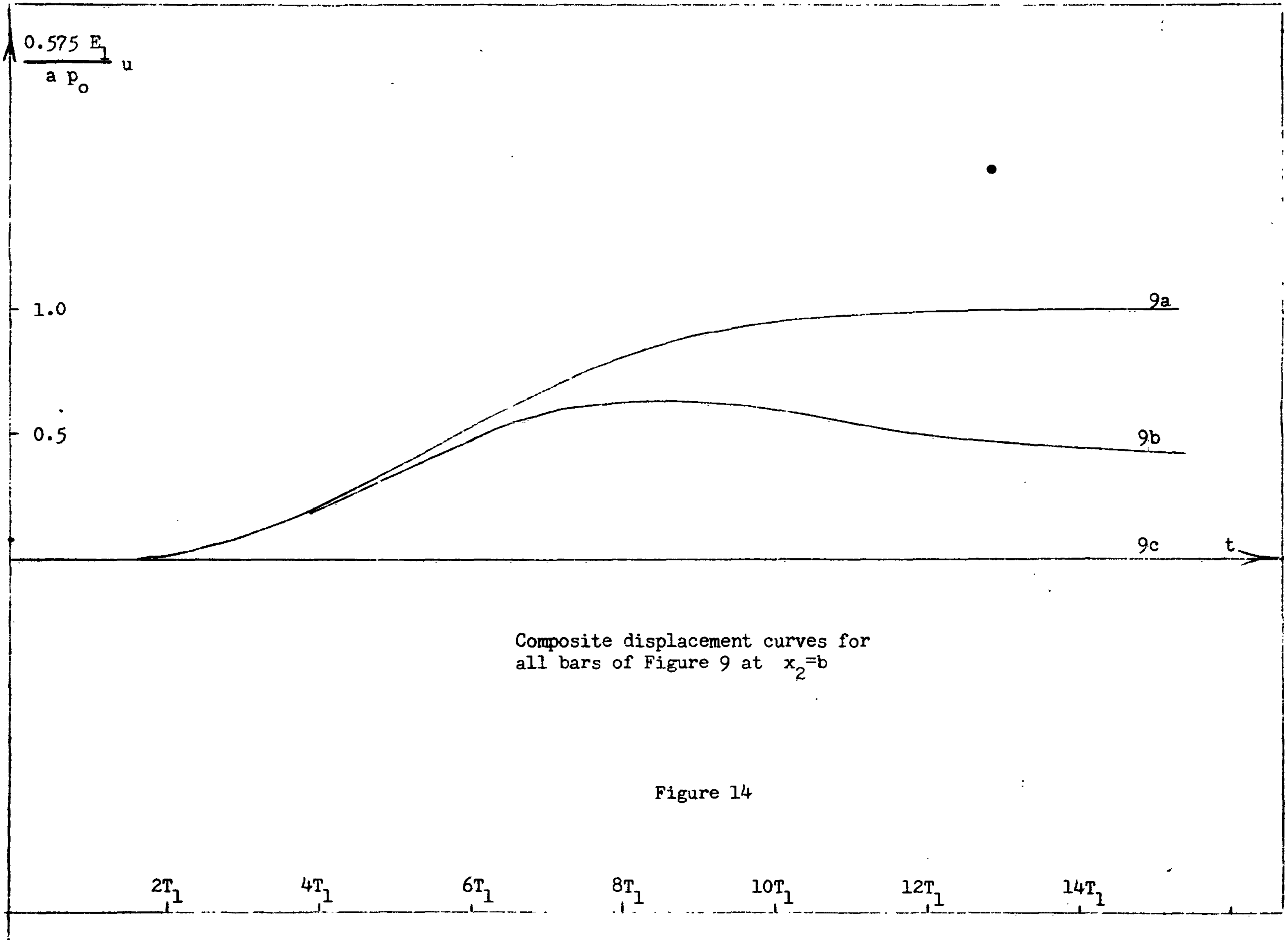
Composite displacement curve for all bars of Figure 9 at $x_1 = a$

Figure 12



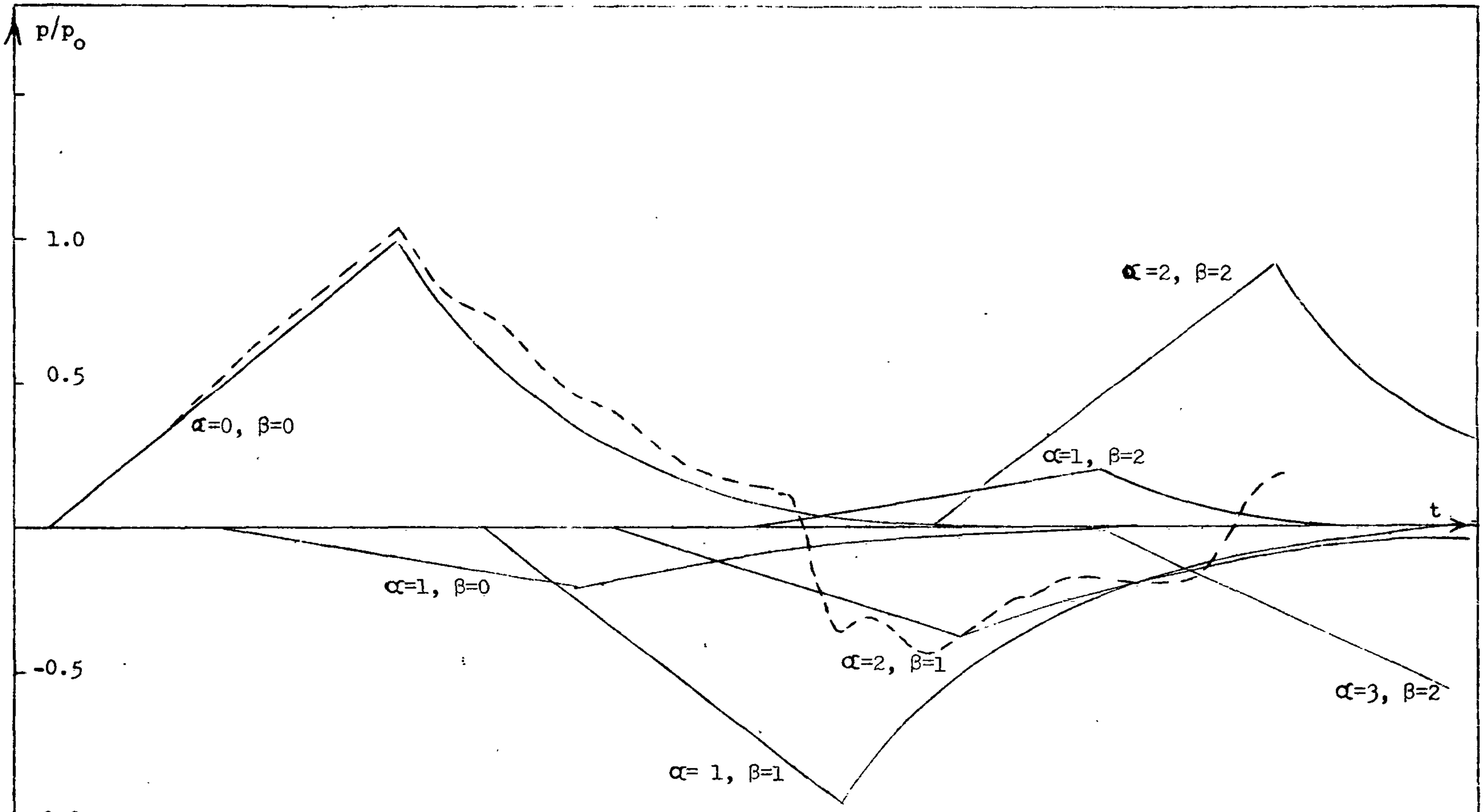
Composite displacement curves for all bars of Figure 9 at $x_2 = b/3$

Figure 13



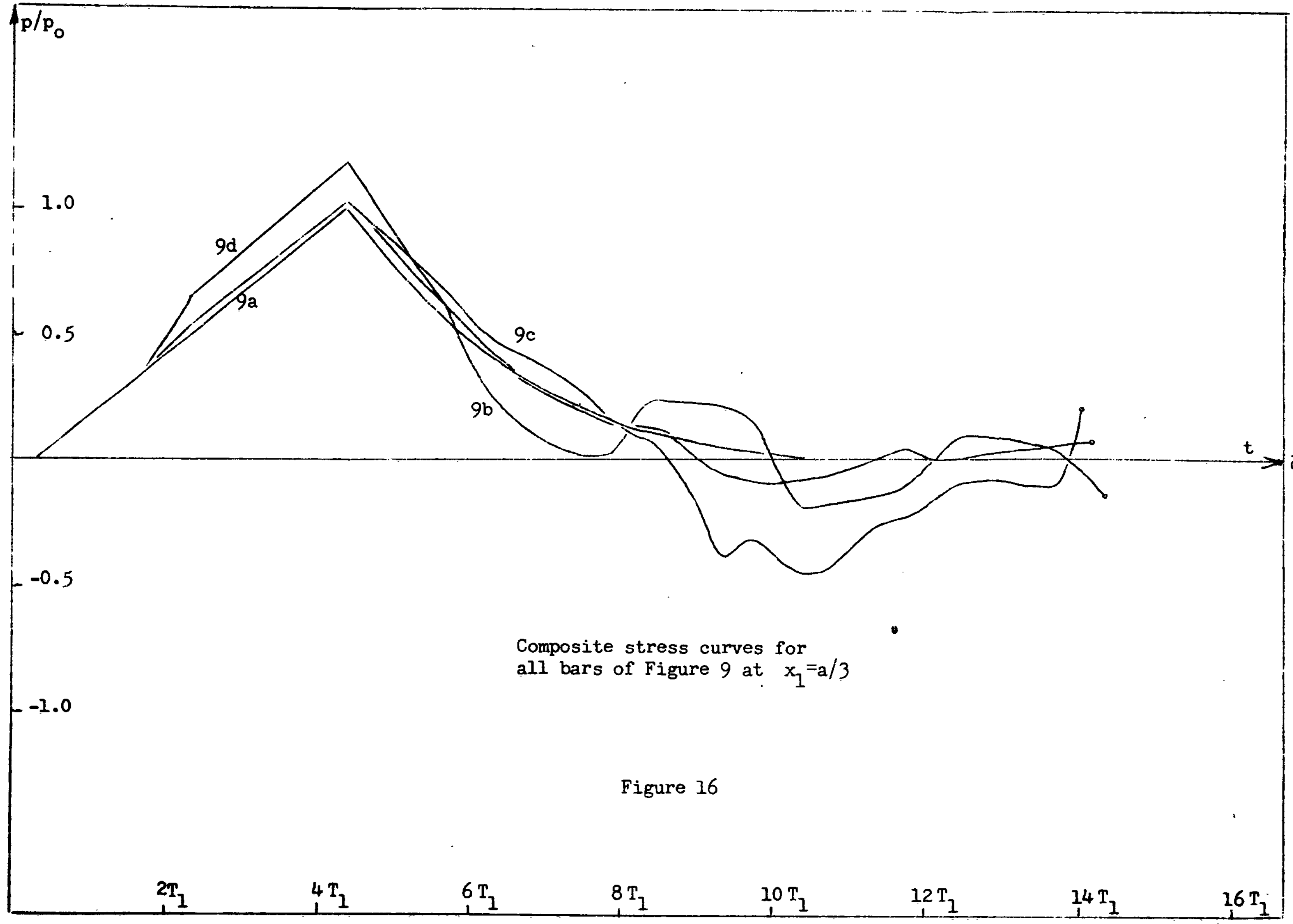
Composite displacement curves for all bars of Figure 9 at $x_2=b$

Figure 14



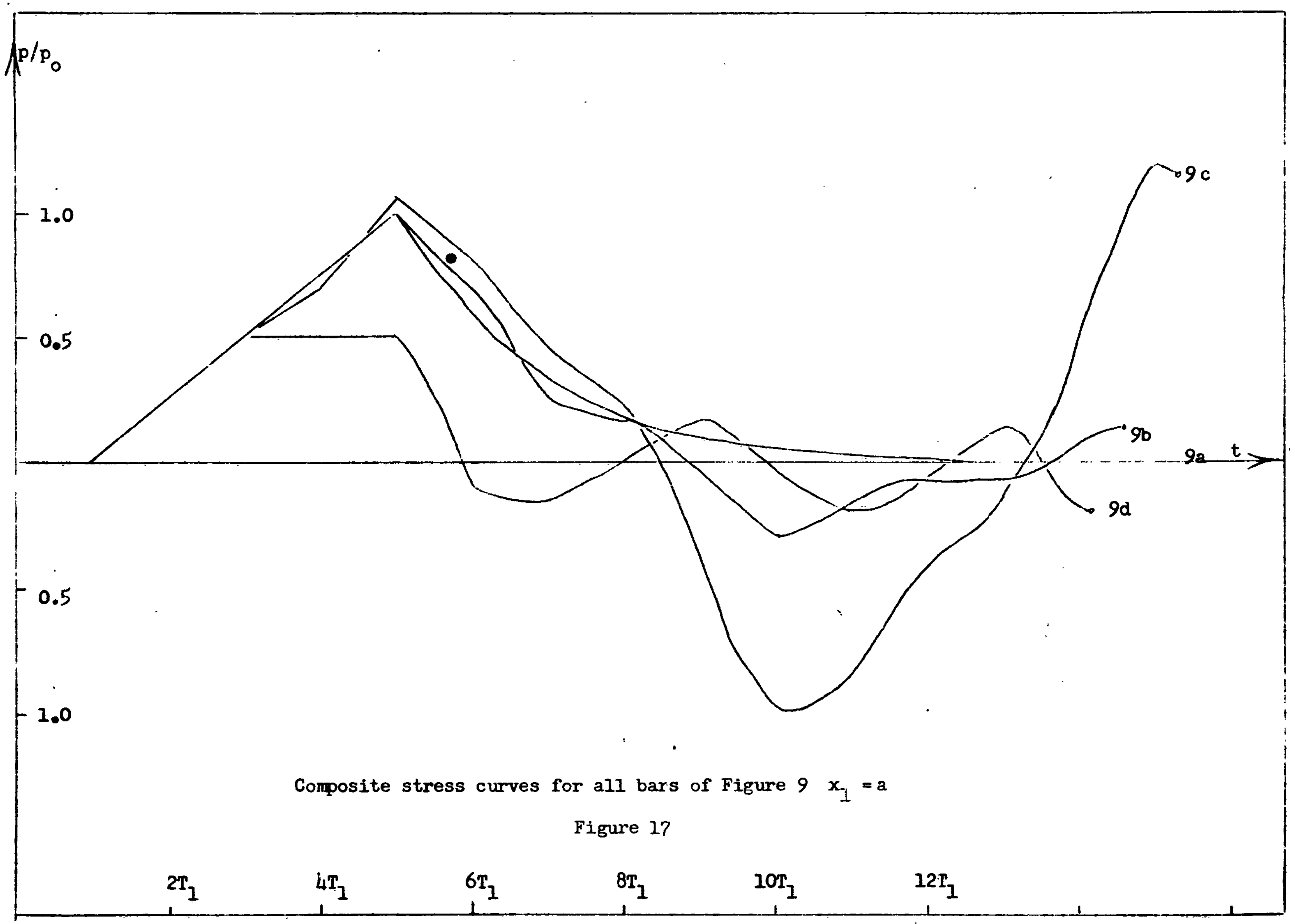
Stresses for bar of Figure 9c at $x_1 = a/3$

Figure 15



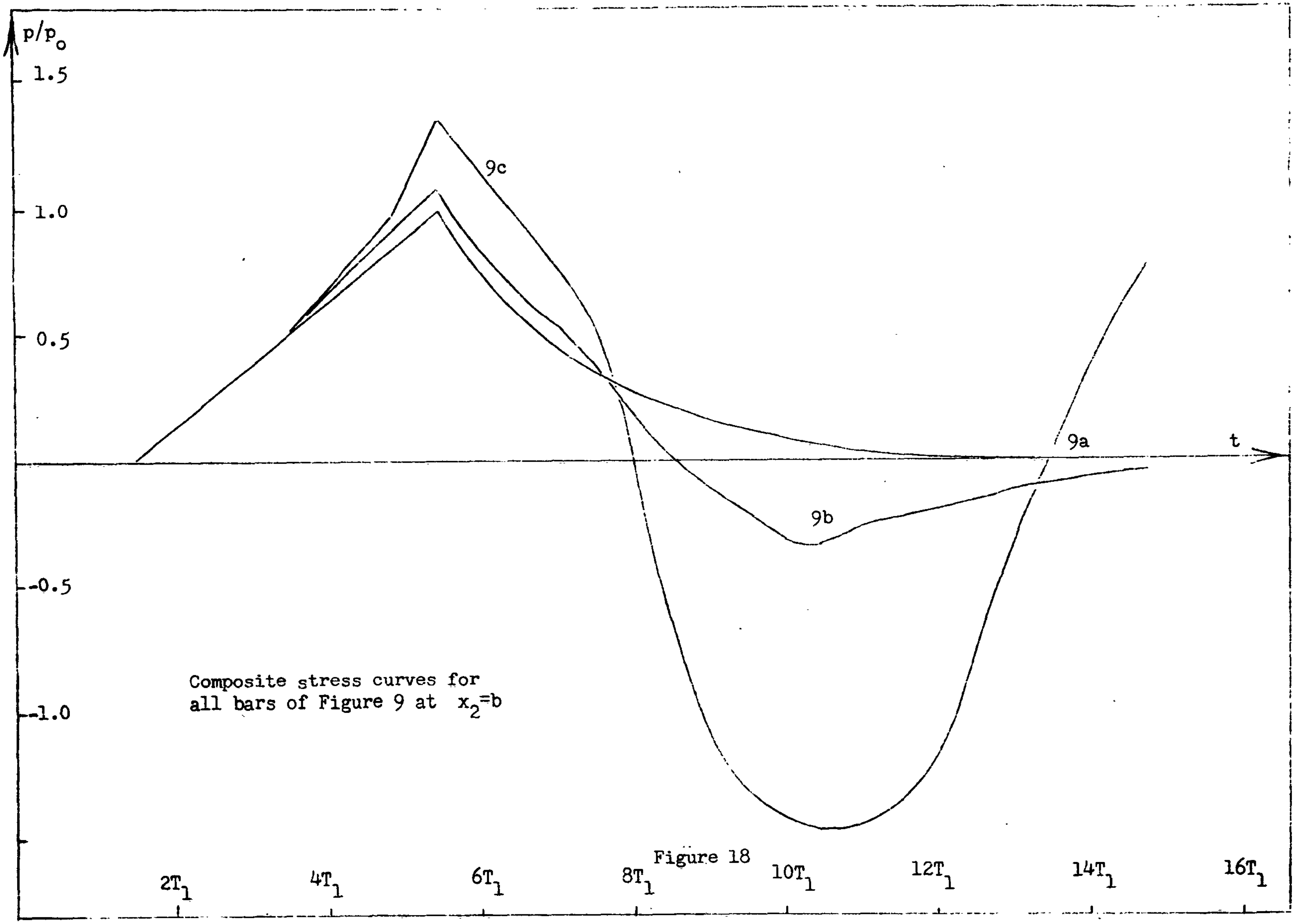
Composite stress curves for all bars of Figure 9 at $x_1 = a/3$

Figure 16



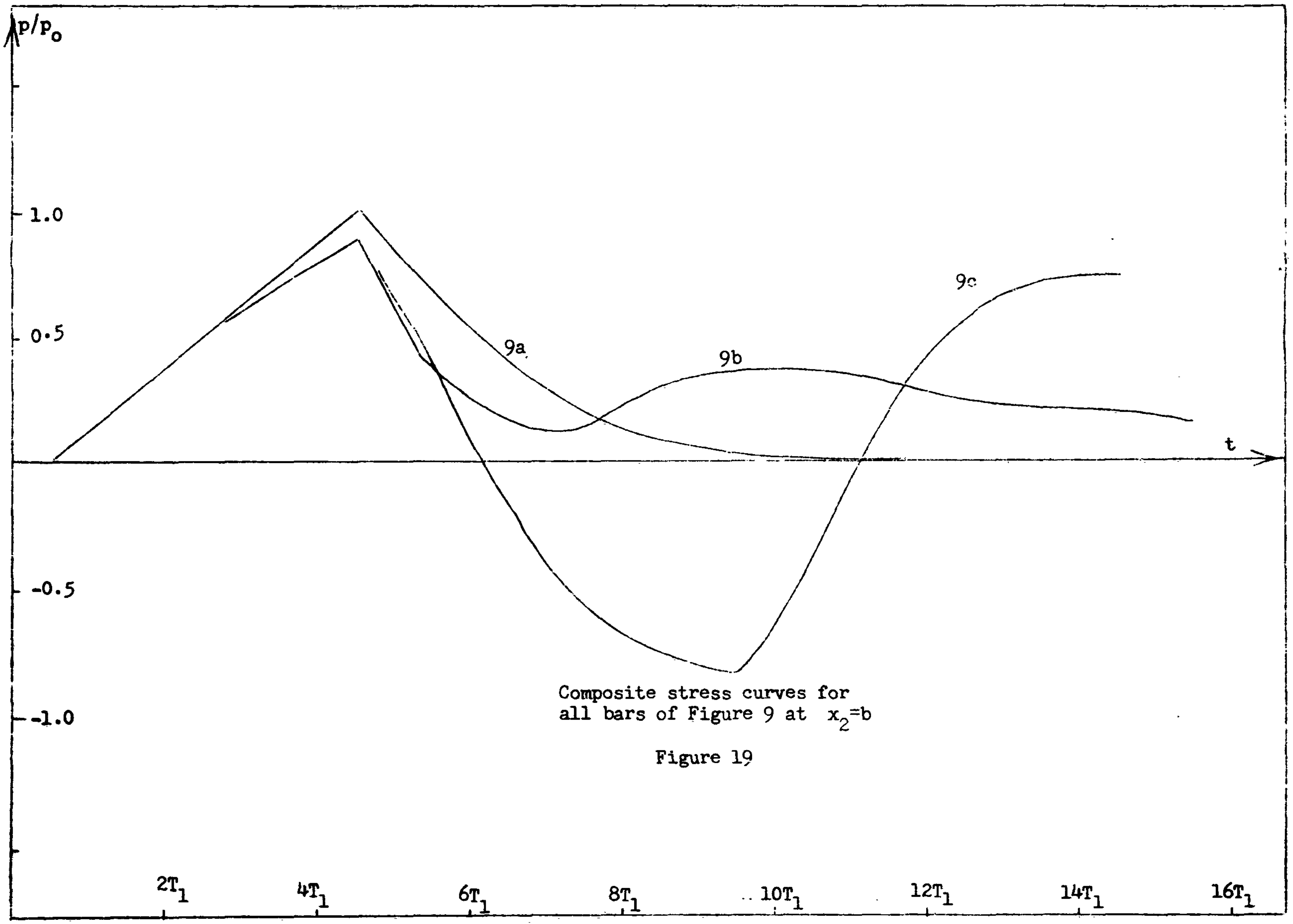
Composite stress curves for all bars of Figure 9 $x_1 = a$

Figure 17



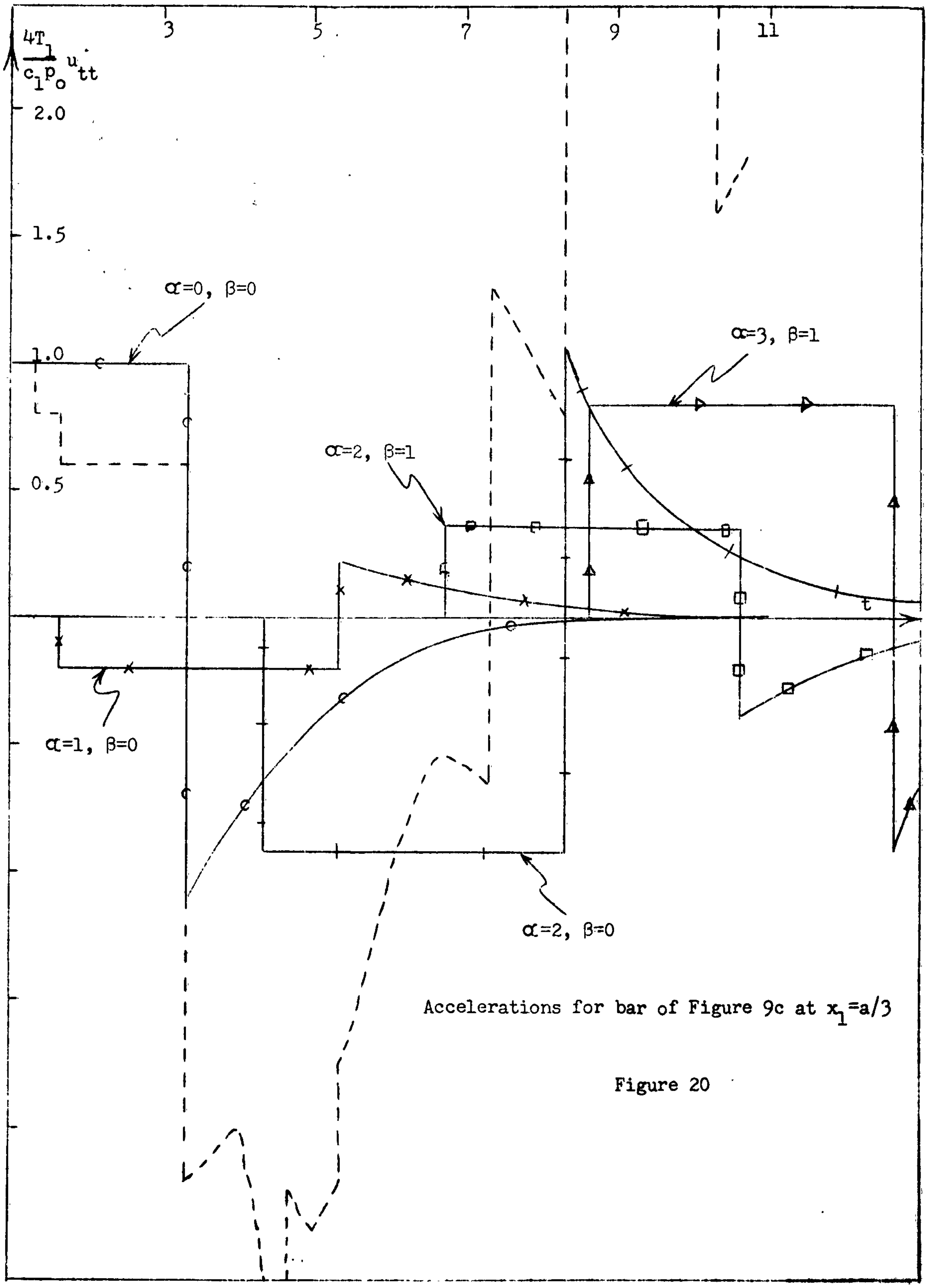
Composite stress curves for all bars of Figure 9 at $x_2=b$

Figure 18



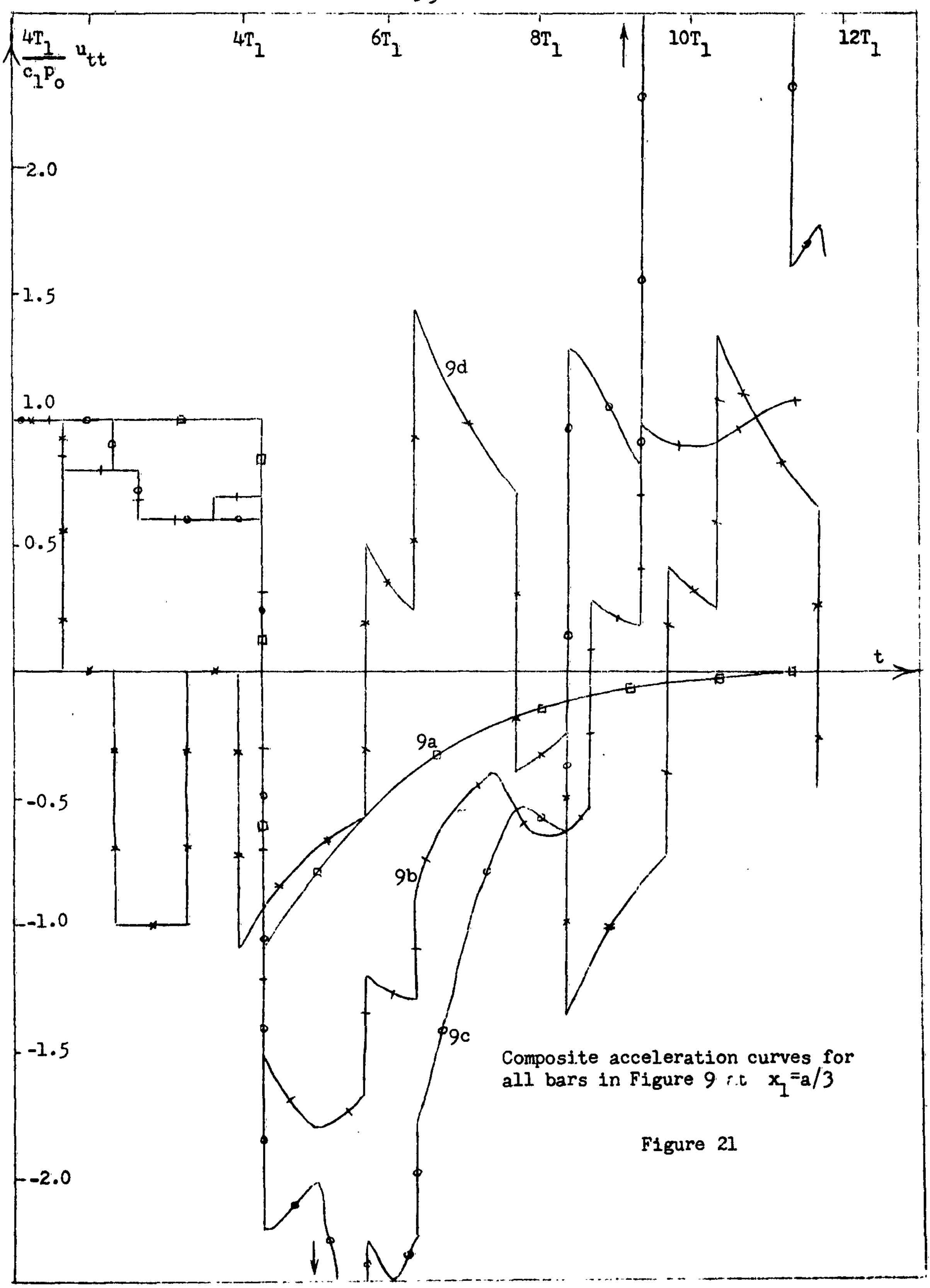
Composite stress curves for all bars of Figure 9 at x_2=b

Figure 19



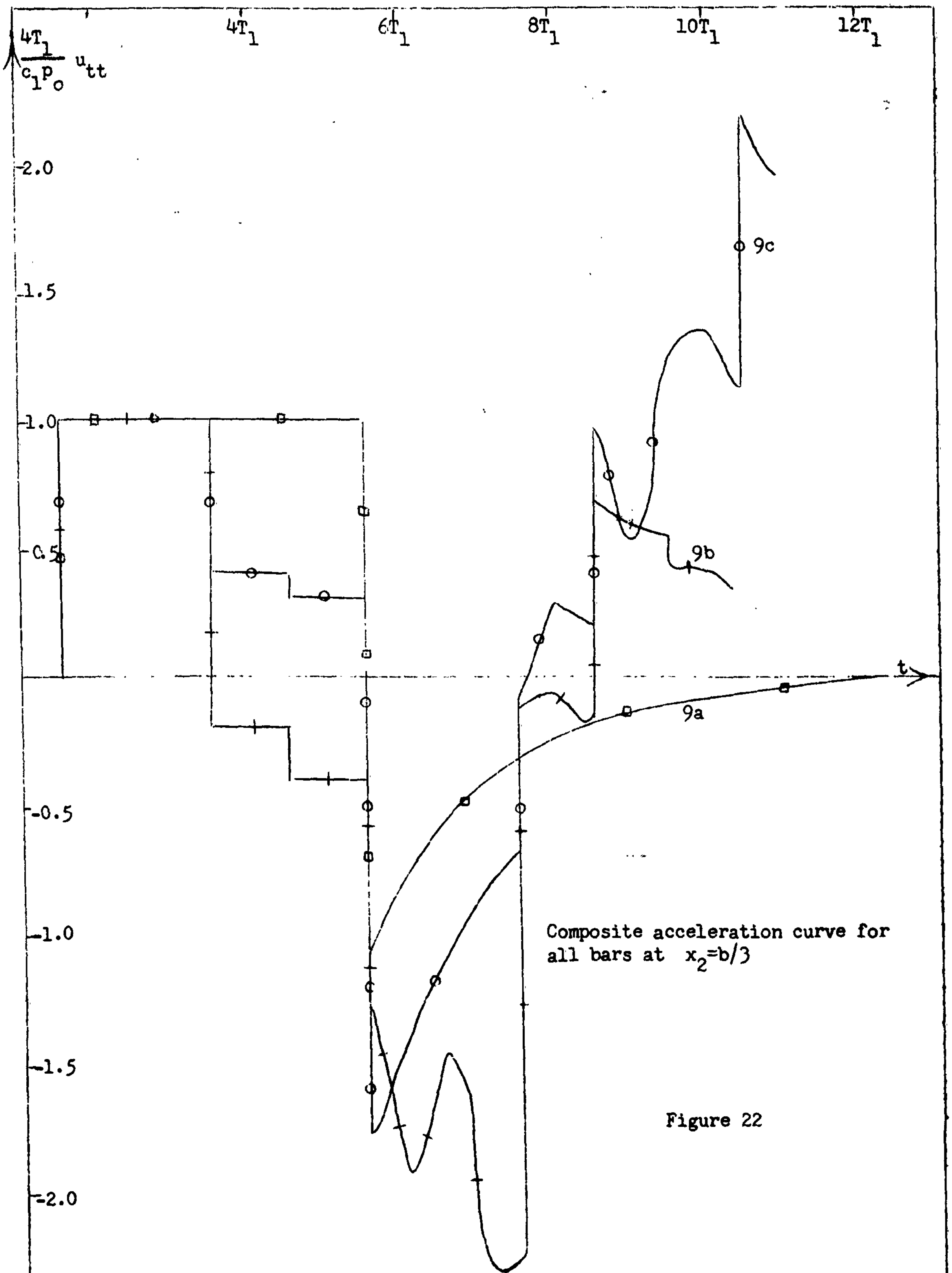
Accelerations for bar of Figure 9c at x_1=a/3

Figure 20



Composite acceleration curves for all bars in Figure 9 at $x_1=a/3$

Figure 21



Composite acceleration curve for all bars at $x_2=b/3$

Figure 22